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A GENERALIZATION OF SIERPIŃSKI'S THEOREM ON STEINER TRIPLES AND THE AXIOM OF CHOICE

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In the language of combinatorial analysis, a finite set F is said to possess a *Steiner triple system* if and only if there exists a family $\mathcal{J}$ of subsets of F such that 1) each element of $\mathcal{J}$ contains exactly three elements of F and 2) every subset of F, containing exactly two elements, is contained in exactly one of the elements of $\mathcal{J}$. It has been long established that a necessary and sufficient condition for the existence of such a system for a finite set F is that $\overline{\overline{F}} \equiv 1$ or $3 \pmod 6$.

In [1], W. Sierpiński has showed that a Steiner triple system always exists for any set which is not finite. The proof of this result depends upon the axiom of choice. In [2], B. Sobociński has proved that the assumption that every non-finite set possesses a Steiner triple system is, in fact, equivalent to the axiom of choice.

The aim of the present paper is to further generalize these two results. We begin by making a

Definition 1: *An arbitrary set E is said to possess a Steiner system of order k (where k is a natural number >1) if there exists a family $\mathcal{J}_k$ of subsets of E such that 1) each element of $\mathcal{J}_k$ contains exactly k elements of E and 2) every subset of E, containing exactly k-1 elements, is contained in exactly one member of the family $\mathcal{J}_k$.*

§1. With the aid of the axiom of choice we shall show that every set which is not finite possesses a Steiner system of order n for $n = 2,3,4,\ldots$. In addition, we shall establish that the assumption that every set which is not finite possesses a Steiner system of order n, for $n = 3,4,\ldots$, is equivalent to the axiom of choice. We are not able to demonstrate the necessity of the axiom of choice to establish the existence of a Steiner system of order *2* for any set which is not finite.

To this end we first prove, with the aid of the axiom of choice,

Theorem 1: *Let E be any set which is not finite. Then E possesses a Steiner system of order n for $n = 3,4,\ldots$.*

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