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AXIOMATISATIONS OF THE MODAL CALCULUS Q

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R. A. Bull has shown in [1] that the modal calculus **Q** of [2] may be axiomatised by taking as primitives a strong and a weak necessity $\mathbf{L}$ and L, and by adding to **PC** the axioms

A1. $CLpp$
A2. $C\mathbf{L}pp$
A3. $CK\mathbf{L}p\mathbf{L}q\mathbf{L}Kpq$

and the rules (beside substitution and detachment)

RQLa: $\vdash C\beta\gamma \rightarrow \vdash C\beta L\gamma$, for β fully modalised and with all its variables occurring in γ.
RQLb: $\vdash C\mathbf{L}\alpha C\beta\gamma \rightarrow \vdash C\mathbf{L}\alpha C\beta L\gamma$, for β fully modalised and with all its variables occurring in α or γ.
RQL: $\vdash C\mathbf{L}\alpha C\beta\gamma \rightarrow \vdash C\mathbf{L}\alpha C\beta\mathbf{L}\gamma$, for β fully modalised and with all variables of β and γ occurring in α.

From the sufficiency of these postulates it is possible to prove the sufficiency of some other postulates for **Q** which I suggest in [3]. In these, I adopt a suggestion of J. L. Mackie and use as a primitive a functor S ("always statable"), such that Sp is equivalent, in terms of Bull's primitives, to $\mathbf{L}Cpp$. The other primitive I use in [3] is a possibility-operator M (in Bull's terms NLN), but Bull's weak necessity L will do just as well, and indeed makes possible a slight simplification of the postulates. Bull's $\mathbf{L}p$ is definable in terms of my primitives as $KSpLp$. My postulates, for subjoining to **PC**, then become the one axiom *A1.* $CLpp$, and the three rules:—

RS1: $\vdash CS\alpha Sp$, where p is any variable in α.
RS2: $\vdash CSpCSq \ldots\ldots S\alpha$, where p, q, etc. are all the variables in α.
RSL: $\vdash C\alpha\beta \rightarrow \vdash CSpCSq \ldots\ldots C\alpha L\beta$, where α is fully modalised and p, q, etc. are all the variables in β that are not in α.

In view of Bull's result, the sufficiency of these for **Q** may be shown by deducing Bull's postulates from them, including a pair of implications ($CSp\mathbf{L}Cpp$ and $C\mathbf{L}CppSp$) corresponding to the definition of S in Bull's system.

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