Notre Dame Journal of Formal Logic
Volume V, Number 3, July 1964

TYPES IN COMBINATORY LOGIC[1]

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The introduction of types in a formal system amounts to giving a classification of the entities of the system in categories, in such a way that the category of complex entities is determined by the categories of others that are simpler. In applications this procedure is used to define the class of formal entities that are propositions, i.e., those for which the rules of the propositional and predicate calculus are valid. In most systems the rules governing types are trivial, and it seems there is little interest in the study of those rules in a more general setting. In combinatory logic the situation has evolved in a different way. Curry has studied in several papers a theory of functionality.[2] In this system the combinators are allowed to belong to many distinct types; this is an important difference from ordinary type theories in which an entity belongs at most to one type. As a consequence the theory is not trivial, and very elaborate arguments are used in [7] to prove some fundamental properties.

In this paper the results of [7] are extended in several directions. For this purpose some properties of pure combinatory logic are necessary. Some of them are available in [7] or [9]; others are new, and the proofs are given in detail.

1. *The system* Γ. We shall consider a system of combinatory logic in which there are three primitive combinators: **S**, **K** and **I**, and possibly other atoms.[3] The atoms that are not combinators are called indeterminates. Combinations are the following formal entities: the primitive combinators and indeterminates are combinations; if X and Y are combinations, then the ordered pair consisting of X and Y in that order is also a combination which is denoted (XY).[4] When writing expressions denoting combinations we shall omit certain parentheses with the understanding that the association is to the left; parentheses at the end and at the beginning of an expression are also omitted. In this way any combination can be expressed in a unique way in the form $X_0 \dots X_t$ $(t \geq 0)$, where X_0 is a primitive atom, called the *leading atom* of the combination. This can be shown easily by induction on the structure of the combination.

Sometimes we shall say that a combination U is a part of a combination X and that Y is obtained by replacing the part U by the combination V. We

Received April 22, 1963