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MODAL SYSTEMS IN THE NEIGHBOURHOOD OF T

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Unpublished is the result of B. Sobociński that if we form systems T_n by adjoining to Feys's modal system T the axiom P_n: $CL^n pL^{n+1}p$ where L^n denotes a string of n L-s $(n \geq 0)$, then while obviously T_n contains T_{n+1}, the converse is not the case. Hence there are infinitely many systems between $S4 = T_1$ and T. We now ask whether the addition of B_1: $LCpLNLNp$ (Lewis's C12) to T_n and T, producing T_n^+ and T^+, similarly yields infinitely many systems between $S5 = T_1^+$ and T^+, and show that this is so. Further, let $S1_n^+$ be the $S1_1^+$ of [1] augmented by P_n. Clearly $S1_1^+ = T_1^+ = S5$, while the matrix used in [2] *ad* 2 shows that if $n > 1$, $S1_n^+$ is a proper subsystem of T_n. Evidently, $S1_n^+$ contains $S1_{n+1}^+$. If $S1_n^+$ and $S1_{n+1}^+$ were equivalent, the addition to each of $LCpMp$ would produce equivalent systems; but these would be T_n^+ and T_{n+1}^+ which are not equivalent. Hence $S1_{n+1}^+$ is a proper subsystem of $S1_n^+$. Since infinitely many reductions of modality thus fail in T^+, T and $S1^+$, these all have infinitely many non-equivalent modalities, as has long been known for T.

To prove that for all n, T_n^+ is independent of T_{n+1}^+, we interpret T_n in the domain of $n+1$-sequences each place of which is filled by *1* or *2*. We base the systems on N (negation), L (necessity) and C (implication). If F is N or L, $F(x_1, \ldots, x_{n+1}) = Fx_1, \ldots, Fx_{n+1}$. $Nx_i = 1$ if $x_i = 2$, $Nx_i = 2$ *if* $x_i = 1$. $Lx_i = 2$ if $x_{i-1} = 2$ or $x_i = 2$ or $x_{i+1} = 2$; *otherwise* $Lx_i = 1$. $C(x_1, x_2, \ldots, x_{n+1})(y_1, y_2, \ldots, y_{n+1}) = Cx_1y_1, Cx_2y_2, \ldots, Cx_{n+1}y_{n+1}$; $Cx_iy_i = 2$ *if* $x_i = 1$, $y_i = 2$, otherwise $Cx_iy_i = 1$. A sequence consisting only of *1*-*s* is designated. The reader may like to compare our version of L with those discussed in [3], pp. 23-4 and [4].

Convenient axioms and rules of T are I. Propositional calculus with rules of substitution, detachment and definition applied to II. M = def. NLN; III. From α infer $L\alpha$; *A1* $CLpp$; *A2* $CLCpqCLpLq$. For T^+ we add *A3* $CpLNLNp$, and for T_n^+, *A4* $CL^n pL^{n+1}p$. The method of valuation obviously satisfies I.

Ad III: If no valuation of α contains 2, the same is true of $L\alpha$.

Ad *A1*: For any valuation of p, $Lp = 1$ at any position only if $p = 1$ at that position.

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