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A REMARK CONCERNING THE THIRD THEOREM ABOUT THE EXISTENCE OF SUCCESSORS OF CARDINALS

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The following three formulas about the existence of successors of cardinals:

S_1 *For every cardinal* $\mathfrak{m}$ *there is a cardinal* $\mathfrak{n}$ *such that* (i) $\mathfrak{m} < \mathfrak{n}$, *and* (ii) *the formula* $\mathfrak{m} < \mathfrak{p} < \mathfrak{n}$ *does not hold for any cardinal* $\mathfrak{p}$.

S_2 *For every cardinal* $\mathfrak{m}$ *there is a cardinal* $\mathfrak{n}$ *such that* (i) $\mathfrak{m} < \mathfrak{n}$, *and* (ii) *for every cardinal* $\mathfrak{p}$ *the formula* $\mathfrak{m} < \mathfrak{p}$ *implies* $\mathfrak{n} \leq \mathfrak{p}$.

S_3 *For every cardinal* $\mathfrak{m}$ *there is a cardinal* $\mathfrak{n}$ *such that* (i) $\mathfrak{m} < \mathfrak{n}$, *and* (ii) *for every cardinal* $\mathfrak{p}$ *the formula* $\mathfrak{p} < \mathfrak{n}$ *implies* $\mathfrak{p} \leq \mathfrak{m}$.

are discussed by Tarski in [2] who has shown there that S_1 can be proved without the help of the axiom of choice and that S_2 is equivalent to this axiom. Concerning S_3 it is remarked in [2], p. 32, that it is not yet known whether S_3 can be proved without the help of the axiom of choice, and, therefore, *a fortiori* it is not known whether S_3 is equivalent to the said axiom. The latter problem remains open, but according to the announcement given in [1], p. 73, note 2, the former one is solved in the negative by A. Lewi who has proved that S_3 does not follow from the axioms of the general set theory, even if the ordering principle is added to these axioms.[1] As far as I know this result of Mr. Lewi is not yet published.

In this note I show that each of the given below formulas, T_1 and T_2, is such that the axiom of choice follows from it and S_3. The formulas T_1 and T_2 are, as I conjecture, probably neither provable without the aid of the axiom of choice nor equivalent to this axiom.

In order to present the formulas T_1 and T_2 and the subsequent deductions in a more compact way I introduce here the following abbreviative definition:

D1 *For any* $\mathfrak{m}$ *and* $\mathfrak{n}$, $\mathfrak{m} \prec \mathfrak{n}$ *if and only if* $\mathfrak{m}$ *and* $\mathfrak{n}$ *are cardinals*, $\mathfrak{m} < \mathfrak{n}$, *and for every cardinal* $\mathfrak{p}$ *the formula* $\mathfrak{p} < \mathfrak{n}$ *implies* $\mathfrak{p} \leq \mathfrak{m}$.

Using this definition we can present T_1 and T_2 as follows:

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