Notre Dame Journal of Formal Logic
Volume III, Number 3, July 1962 

# RECURSIVE LINEAR ORDERINGS AND HYPERARITHMETICAL FUNCTIONS

SHIH-CHAO LIU

The main purpose of this note is to give an alternative proof to a theorem by Spector [1] which answers a question raised by Kleene [3, p. 25]. There are two by-products. The first (Theorem 1) specifies a sufficient condition for a set linearly ordered by a recursive ordering to have a well-ordered segment of a certain order type.[1] The second (Theorem 2) is a generalization, in some sense, of a theorem of Kleene [4, XXVL]. This enables us to apply Kleene's [3, Theorem 2] directly in our proof of Spector's theorem (Theorem 3 in this note). So it seems that the proof becomes much shorter.[2]

We first introduce some notations. $f \in \mathbf{L} \equiv \{f$ is a Gödel number of some recursive linear ordering $\overset{f}{\prec}$ which orders some set $M_f\}$ [2]. $f \in \mathbf{W} \equiv \{f \in \mathbf{L}$ & $M_f$ is well-ordered by $\overset{f}{\prec}\}$ [2]. $\mathbf{S}(f, n)$ is a primitive recursive function such that $f \in \mathbf{L}$ implies (i) $\mathbf{S}(f, n) \in \mathbf{L}$ for all $n$, (ii) if $n \notin M_f$, $M_{\mathbf{S}(f, n)}$ is empty, (iii) if $n \in M_f$, $M_{\mathbf{S}(f, n)}$ is a segment $\hat{x}(x \overset{f}{\prec} n)$ of $M_f$ and $x \overset{\mathbf{S}(f, n)}{\prec} y \equiv x \overset{f}{\prec} y$ for all $x, y \in M_{\mathbf{S}(f, n)}$ [2, p. 156]. $\|f\|$ is the order type of $\overset{f}{\prec}$ if $f \in \mathbf{L}$, $|b|$ is the order type named by $b$, if $b \in 0$ [2]. $y^*$ stands for $2^y$, $H_y(u)$ is defined as in [2].

*Theorem 1.* If $f \in \mathbf{L}$, $f \notin \mathbf{W}$, $y \in 0$ and for every function $\alpha(i)$ recursive in $H_{y^{**}}$, (i) $(\alpha(i+1) \overset{f}{\prec} (i))$, then for every $b \in 0$ with $|b| < |y|$, there is some $n \in M_f$ such that $|b| = \|\mathbf{S}(f, n)\|$.

*Proof* (by induction on the ordinal $|b|$). The proof for the case $|b| = 0$ is simple.

Suppose $0 < |b| < |y|$. Let enm$(i)$ be a primitive recursive function which enumerates all the numbers $<_0 b$ [6]. By the induction hypothesis, for every $i$, there is some $n_i \in M_f$ such that $|\text{enm}(i)| = \|\mathbf{S}(f, n_i)\|$. Let $n_i$ be determined as a total function of $i$ by $n_i = \mu z(z \in M_f \;\&\; |\text{enm}(i)| = \|\mathbf{S}(f, z)\|)$. Note that $|\text{enm}(i)^{**}| \leqq |b^*| \leqq |y|$ we see that $n_i$ is recursive in $H_y$ by [2, Theorem 3 and Theorem 5].

Since $\mathbf{S}(f, n_i) \in \mathbf{W}$ for every $i$ and by the supposition of the theorem,

*Received September 29, 1961*