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A SIMPLE PROOF OF FUNCTIONAL COMPLETENESS IN MANY-VALUED LOGICS BASED ON ŁUKASIEWICZ'S *C* AND *N*

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Past investigations, [1], [2] and [3], have used the integers $1, 2, \ldots, n$ as truth-values for an n-valued logic. In such a logic, the truth-functions associated with C and N have the following definitions

$$C(p, q) = \max(1, q - p + 1); \qquad N(p) = n - p + 1.$$

Here we shall use $n + 1$ - valued logics with truth-values $0, 1, \ldots, n$. As a result, the above definitions simplify to

$$C(p, q) = \max(0, q - p); \qquad N(p) = n - p.$$

Not only does this simplify the computations involved, but also makes a simple line of proof apparent. No logical tools are used, and the only non-trivial number-theoretic result used is "If $(a, b) = d$,[1] then there are integers x and y for which $ax + by = d$."

Theorem 1. Any function[2] which takes the value 0 once and n otherwise is generated by C and N.

1. $C(p, p) = 0$.
2. $N(0) = n$.
3. $\alpha_m(p_1, \ldots, p_m) = \min(n, p_1 + p_2 + \ldots + p_m)$ is generated for $m \geq 1$.

   Proof is by induction.

   $C(0, p_1) = p_1 = \min(n, p_1) = \alpha_1(p_1)$.

   Suppose that α_k is generated for $k \geq 1$.

   $N(\alpha_k(p_1, \ldots, p_k)) = \max(0, n - (p_1 + \ldots + p_k))$.

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[1] $(a, b) = d$ means that d is the greatest common divisor of a and b.

[2] All functions used in this paper will have $0, 1, \ldots, n$ as the domain for each argument and will take values in this set.

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