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PROOF OF SOME THEOREMS ON RECURSIVELY ENUMERABLE SETS

TH. SKOLEM

In this paper I shall first define a class of functions which I call lower elementary, abbreviated l.el. functions in the sequel, and after some preliminary considerations prove that every recursively enumerable set of integers can be enumerated by a l.el. function. All variables and functions shall here take non-negative integers as values. L. Kalmár defined the notion elementary function (see [1]). These are the functions that can be constructed from addition, multiplication and the operation $\dot{-}$ by use of the general sums and products

$$\sum_{r=o}^{x} f(r) \quad \text{and} \quad \prod_{r=o}^{x} f(r),$$

where f may contain parameters, together with the use of composition. If we omit the use of general products, we get what I call the lower elementary functions. The definition is therefore:

Df 1. The l.el. functions are those which can be built by starting with the functions 0, 1, $x + y$, xy, $x \dot{-} y$ and using the summation $\sum_{r=o}^{x} f(r)$, where f may contain parameters, besides use of composition. By the way, instead of $x \dot{-} y$ one can choose $\delta(x, y)$, the Kronecker delta (see [2]). As to the summation schema it can be shown that it is sufficient to require its use in the case that f contains one parameter at most. Of course xy can be omitted as starting function.

Clearly every polynomial is an l.el. function. Further every l.el. function can be majorised by a polynomial. This is seen immediately to be true for the starting functions and it is easily seen to be hereditary with regard to summation and composition. If for example $f(x, y)$ is always $\leqq \varphi(x, y)$, where φ is a polynomial, then for all x and y

$$\sum_{r=o}^{x} f(r, y) \leqq \sum_{r=o}^{x} \varphi(r, y)$$

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