A NOTE CONCERNING THE MANY-VALUED PROPOSITIONAL CALCULI

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For the purpose of investigating many-valued propositional calculi the following simple theorem is often useful:

THEOREM: *Any implicational thesis of the bi-valued propositional calculus in which only one variable occurs is a thesis in a many-valued system* $\mathfrak{S}$ *provided the following conditions are satisfied:* (i) *implication is primitive or defined in* $\mathfrak{S}$; (ii) *the ordinary operations of substitution and detachment in respect of implication are valid in* $\mathfrak{S}$; (iii) *any implicational thesis of* $\mathfrak{S}$ *is a thesis of the bi-valued calculus, and* (iv) *the following theses hold in* $\mathfrak{S}$: *A1 Cpp; A2 CpCCqrCCpqr or A2* CCqrCpCCpqr; A3 CCrpCCqpCrCqp or A3* CCqpCCrpCrCqp; A4 CCprCCqpCCrpCqp or A4* CCqpCCprCCrpCqp; A5 CCprCCqpCqCCrpp or A5* CCqpCCprCqCCrpp.*

PROOF: Assume that **F** is an arbitrary implicational bi-valued thesis in which there occurs only one variable, say "p". Then the last consequent of **F** is equiform with "p" and this variable is preceded by n (for: $1 \leq n < \infty$) antecedents. Hence, **F** is either **TI**: $C\alpha_1 C\alpha_2 \ldots C\alpha_{n-1} C\alpha_n p$ or **TII**: $C\alpha p$. If **F** is **TI**, then there must exist such α_k (for: $1 \leq k \leq n$) that formula **G**: $C\alpha_k p$ which is shorter than **F** and is **TII** itself, is a thesis of the classical propositional calculus. Otherwise, **F** could not be a thesis of $\mathfrak{S}$. Consider now a formula α_j which is a part of **F** and is such that $j \neq k$ and $1 \leq j \leq n$. If α_j is false, then $C\alpha_j p$ is a thesis of the bi-valued propositional calculus, and together with **G** and *A3* (or *A3**) implies theses: *B1* $C\alpha_j C\alpha_k p$ and *B2* $C\alpha_k C\alpha_j p$. On the other hand, if α_j is a thesis, then its form is: $C\beta_1 C\beta_2 \ldots C\beta_{m-1} C\beta_m p$, for: $1 \leq m < \infty$. Since this is an implicational thesis, the following formula: $CpC\beta_1 C\beta_2 \ldots C\beta_{m-1}\beta_m$, to be referred to as *C1*, is of the same length, and is also a thesis of the bi-valued propositional calculus. In such a case *C1*, **G**, *A4* (or *A4**) and *A5* (or *A5**) imply *B1* and *B2* again. Therefore, elementary inductive reasoning shows that if **F** is **TI**, then it can always be obtained from *A3*, *A4*, *A5* (or *A3**, *A4**, *A5**), and shorter theses in which only one variable occurs. Consequently, any thesis of type **TI** can be reduced to shorter theses of type **TII**. But, any thesis of this type is either *A1* or has the following form: *D1* $CC\gamma_1 C\gamma_2 \ldots C\gamma_{r-1} C\gamma_r pp$, for: $1 \leq r < \infty$. Since *D1* is a thesis, then any of its components γ_j, for: $1 \leq j \leq r$, must also be a thesis of the bi-valued calculus. Hence *D1* follows from *A1*, *A2* (or *A2**) and the theses $\gamma_1, \gamma_2, \ldots \gamma_r$ each of which is shorter than *D1*. Since any thesis γ_j is of type **TI** or **TII**, an elementary proof by induction

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