ON RECURSIVE TRANSCENDENCE

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1. Let $P_n(x)$ be the n^{th} polynomial in an enumeration of all one-variable polynomials with integral coefficients; let $||z|| = ||x + iy|| = |x| + |y|$ be called the *norm* of a rational complex number $z = x + iy$ and let $\{s_n\}$ be a sequence of rational real or complex numbers. Then $\lim s_n$ is transcendental if

$$(r)(\exists k)(\exists N)(n)\{n \geqslant N \to ||P_r(s_n)|| > 2^{-k}\} \tag{1.1}$$

The convergence of $\{s_n\}$ is expressed by the condition:

$$(k)(\exists \nu)(n)\{n \geqslant \nu \to ||s_n - s_\nu|| < 2^{-(k+2)}\}. \tag{1.2}$$

Let $\nu(k)$ be the least value of ν for which (1.2) holds, so that $n \geqslant \nu(k) \to ||s_n - s_{\nu(k)}|| < 2^{-(k+2)}$, and let k_r and N_r be the least values of k and N for which (1.1) holds, so that

$$n \geqslant N_r \to ||P_r(s_n)|| > 2^{-k_r}. \tag{1.3}$$

Now if $M = \max_{0 \leqslant r \leqslant \nu(1)} \{||s_r|| + 1\}$, and if $P_r^*(x)$ is the sum of the absolute values of the terms of $P_r'(x)$, the first derivative of $P_r(x)$, then

$$||P_r(s_m) - P_r(s_n)|| < ||s_m - s_n||\, P_r^*(M),$$

and, calling the exponent of the least power of 2 which exceeds $P_r^*(M)$, c_r, we have

$$m, n \geqslant \nu(k + c_r) \to ||P_r(s_m) - P_r(s_n)|| < 2^{-k-1}. \tag{1.4}$$

If s_n is general recursive and general recursively convergent, so that the function $\nu(k)$ is general recursive, and if further, the functions N_r and k_r in (1.3) are both general recursive, then the general recursive real (complex) number $\{s_n\}$ is said to be *general recursively transcendental.*

If s_n, $\nu(k)$, N_r and k_r are all primitive recursive (p.r.), then the p.r. real (complex) number $\{s_n\}$ is said to be *primitive recursively* (p.r.) *transcendental.*

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