A NOTE CONCERNING THE AXIOM OF CHOICE

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It is well known[1] that in the set theory the following theorem is provable without the use of the axiom of choice:

I. *If* $\mathfrak{m}$ *is a cardinal number and* $\aleph$ *is an aleph such that* $\aleph \leq \mathfrak{m}$, *then* $\mathfrak{m} = \aleph + \mathfrak{m}$.

It is interesting to note that an analogous formula for the multiplication of cardinals, viz.:

II. *If* $\mathfrak{m}$ *is a cardinal number and* $\aleph$ *is an aleph such that* $\aleph \leq \mathfrak{m}$, then $\mathfrak{m} = \aleph\, \mathfrak{m}$, is equivalent to the axiom of choice.

Proof: It is evident that this axiom implies II. Now, assume II and that $\mathfrak{m}$ is an arbitrary cardinal number which is not finite. Put $\mathfrak{n} = \aleph_0 \mathfrak{m}$. Hence, $\mathfrak{n} = \mathfrak{n} + 1$. We know that for $\mathfrak{n}$ one can construct, without resorting to the axiom of choice a Hartogs' aleph $\aleph(\mathfrak{n})$, i.e. an aleph which is not $\leq \mathfrak{n}$.[2] Since, generally we have $\aleph(\mathfrak{n}) \leq \mathfrak{n} + \aleph(\mathfrak{n})$, then by the application of II we get: $\mathfrak{n} + \aleph(\mathfrak{n}) = \aleph(\mathfrak{n}) \cdot (\mathfrak{n} + \aleph(\mathfrak{n})) = \aleph(\mathfrak{n})\, \mathfrak{n} + \aleph(\mathfrak{n})^2 = \aleph(\mathfrak{n})\, \mathfrak{n} + \aleph(\mathfrak{n}) = \aleph(\mathfrak{n})(\mathfrak{n} + 1) = \aleph(\mathfrak{n})\, \mathfrak{n}$, i.e. we obtain the formula

$$\mathfrak{n} + \aleph(\mathfrak{n}) = \aleph(\mathfrak{n})\, \mathfrak{n}$$

which implies that either $\mathfrak{n} \geq \aleph(\mathfrak{n})$ or $\aleph(\mathfrak{n}) \geq \mathfrak{n}$.[3] Since the first possibility is excluded, we have $\aleph(\mathfrak{n}) \geq \mathfrak{n} = \aleph_0 \mathfrak{m} \geq \mathfrak{m}$. Hence $\mathfrak{m}$ is an aleph and the theorem is proved.

NOTES

[1] Cf. Waclaw Sierpiński: Cardinal and ordinal numbers. *Monografie matematyczne*, tom 34; Warszawa 1958, p. 413, Exercise 1.

[2] Cf. W. Sierpiński, op. cit., pp. 407-409.

[3] Cf. A. Tarski: *Sur quelques théorèmes qui equivalent à l'axiome du choix.* Fundamenta Mathematicae, vol. 5 (1924), pp. 147-154, lemme 1.

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