THE NUMBER OF MODULI IN N-ARY RELATIONS

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As a consequence of theorem 1, p.p. 385-387, in [1], Tarski has proved that two unary relations and four binary relations are definable by purely logical means, and that in general, "for every natural number n only a specifiable finite number of n-termed relations between individuals can be defined by purely logical means, and each of these relations can be expressed by means of identity and the concepts of the sentential calculus." It is the objective of this note to specify the above mentioned number and to exhibit the relations for $n = 3$.

Let n be a fixed natural number and $x_1, x_2, \ldots, x_n$ n individuals.

D1. If $R(x_1, x_2, \ldots, x_n)$ is an n-ary relation definable in terms of identity and the propositional calculus, it is called a modulus.

D2. If $R(x_1, x_2, \ldots, x_n)$ is a modulus which applies to none of the individuals $x_1, x_2, \ldots, x_n$, it is called the empty modulus and is denoted by Φ.

D3. Let $\mathbf{N} = [(i, j) \mid 1 \leq i \leq n, 1 \leq j \leq n]$

D4. If a set S determines a finite number of propositions, $\bigwedge_S P$ denotes the conjunction of these propositions. In some instances the conjunction of $P_1, P_2, \ldots, P_m$, will be denoted by $\bigwedge_{i=1}^{m} P_i$. In like manner, $\bigvee_S P$ will be used for alternation.

Lemma 1. Any non-empty modulus can be expressed in the form $\bigvee_B (\bigwedge_A T)$; where ($\alpha$) $\phi \subsetneqq A \subseteq \mathbf{N}$, ($\beta$) T is either $x_i = x_j$ or $x_i \neq x_j$ for $(i, j) \in A$, (γ) $\bigwedge_A T \neq \Phi$, (δ) $\{\phi\} \neq B$, $\phi \neq B \subseteq P(\mathbf{N})$ where $P(\mathbf{N})$ is the set of subsets of $\mathbf{N}$.

Proof: Any negation sign can be taken into a T by means of $\sim (p \wedge q) . \equiv . \sim p \vee \sim q$ and $\sim (p \vee q) . \equiv . \sim p \wedge \sim q$. Any Φ can be removed by $p \vee \Phi . \equiv p$.

The conditions (α) through (δ) will apply to the next two definitions.

D5. A Kn-form is a proposition of the form $\bigwedge_{\mathbf{N}} T$.

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