A DOUBLE-ITERATION PROPERTY OF BOOLEAN FUNCTIONS

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It is the object of this paper to furnish a proof of a theorem

$$f(x) = f(f(f(x))),$$

which is derivable from the fundamental equation for the expansion of a Boolean function of one variable:

$$f(x) = (f(1) \cap x) \cup (f(0) \cap \overline{x}).$$ [1]

From this proposition we may obtain a simple method for rewriting an iterative Boolean function in terms of a non-iterative Boolean function and also for proving the equivalence of two such functions of one variable.

LEMMA 1. $f(f(1)) = f(0) \cup f(1)$.

PROOF. Using the above fundamental theorem and substituting $f(1)$ for x, we have

$$f(f(1)) = (f(1) \cap f(1)) \cup (f(0) \cap \overline{f(1)})$$

$$= f(1) \cup (f(0) \cap \overline{f(1)}) \qquad [\text{by } x \cap x = x]$$

$$= (f(1) \cup f(0)) \cap (f(1) \cup \overline{f(1)})$$

$$[\text{since } x \cup (y \cap z) = (x \cup y) \cap (x \cup z)]$$

$$= (f(1) \cup f(0)) \cap 1 \qquad [\text{by } x \cup \overline{x} = 1]$$

$$= f(1) \cup f(0). \qquad [\text{since } x \cap 1 = x]$$

LEMMA 2. $f(f(0)) = f(1) \cap f(0)$.

PROOF. Substituting $f(0)$ for x, we have

$$f(f(0)) = (f(1) \cap f(0)) \cup (f(0) \cap \overline{f(0)}) \qquad [\text{by the fundamental theorem}]$$

$$= (f(1) \cap f(0)) \cup 0 \qquad [\text{by } x \cap \overline{x} = 0]$$

$$= f(1) \cap f(0). \qquad [\text{since } x \cup 0 = x]$$

Again applying the fundamental theorem, we can now arrive at equivalent expressions for $f(f(f(0)))$ and $f(f(f(1)))$.

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