AN EXTENSION ALGEBRA AND THE MODAL SYSTEM T

E. J. LEMMON

In [4],[1)] [5], and [6] J.C.C. McKinsey and A. Tarski proved some far-reaching theorems concerning the modal system S4 and its extensions by using techniques of abstract algebra, and in particular the concept of a closure algebra. In [2], M.A.E. Dummett and the present author applied these results to proving the characteristicity of certain matrices for S4 and some of its extensions. In the present paper, a new kind of algebra is introduced, here called an extension algebra, which is shown to have the same utility in studying the modal system T that closure algebras have in the study of S4. Finally, a particular extension algebra is shown to be characteristic for T; this algebra is very similar to the closure algebra shown to be characteristic for S4 in [2]. Acquaintance with the relevant material in [2], [4], [5], and [6] is presumed in what follows, and proofs which model closely their analogues in these papers are omitted.

I

We define an extension algebra as follows:

Definition 1. $\mathfrak{M} = \langle M, \cup, \cap, -, \mathbf{E} \rangle$ is an *extension algebra* iff M is some set of elements and $\cup, \cap, -, \mathbf{E}$ are operations on these elements such that:

(i) M is a Boolean algebra with respect to $\cup$, $\cap$, and $-$;

(ii) if $x \in M$, then $\mathbf{E}x \in M$;

(iii) if $x \in M$, then $x \subseteq \mathbf{E}x$;

(iv) if $x, y \in M$, then $\mathbf{E}(x \cup y) = \mathbf{E}x \cup \mathbf{E}y$;

(v) $\mathbf{E}\Lambda = \Lambda$.

If we compare this definition with [5] Df. 1.1 we see that, if in addition we stipulate that for $x \in M$ $\mathbf{E}\mathbf{E}x = \mathbf{E}x$, $\mathfrak{M}$ is a closure algebra. Thus our definition is a generalization of that of a closure algebra: all closure algebras are extension algebras, but not conversely.

$\mathbf{E}x$ may be called the *extension* of x, and, in analogy with the interior operator of closure algebras, we may define:

Definition 2. For x $\mathbf{E}$ M, $\mathfrak{J}x = -\mathbf{E}-x$.

$\mathfrak{J}x$ is the *intension* of x.

Extension algebras might form the basis of an abstract mathematical study of *growth.* For example, if the elements of M are construed as sets of

Received September 18, 1959