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NOTE ON AN INEQUALITY OF TIBOR RADO

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In his papers [1], [2] and [3] on non-computable functions Tibor Rado has mentioned the inequality $\mathsf{S}(n) \leqslant (n+1)\Sigma(5n)2^{\Sigma(5n)}$, where $\mathsf{S}(n)$ is the maximum number of shifts that can be made by an n card (state) Turing machine—under certain restrictive conditions—and $\Sigma(n)$ is the maximum number of strokes which can be printed by an n card Turing machine. It is the purpose of this note to show that in fact $\mathsf{S}(n) \leqslant (n+1)\Sigma(3n)2^{\Sigma(3n)}$.*

Throughout his papers and, in particular, in defining the functions he is concerned with, Rado uses two-symbol Turing machines which do not have a stay shift i.e., the machine must shift to the right or to the left after each print. The program of each machine is displayed in the form of a set of numbered cards. Each card has two rows of information; the first row having the instructions for the case when the machine scans a blank, the second row having the instructions for the case when the machine scans a stroke. Each row has three pieces of information: the first piece, a *1* or a *0*, determines whether the machine will print a stroke or a blank; the second piece, a *1* or a *0*, determines whether the machine will shift to the right or to the left; and the third piece, a non-negative integer, determines what card will be used for the next set of instructions. If the last number of the set of instructions is *0*, the machine stops. The *1* card is the first card used by the machine.

We are now prepared to define Rado's functions $\Sigma(n)$, $\mathsf{S}(n)$, and the range function $\mathsf{R}(n)$. $\Sigma(n)$ is the maximum number of strokes left by an n card Turing machine starting with a blank tape and stopping after a finite number of shifts. $\mathsf{S}(n)$ is similarly defined as the maximum number of shifts an n card Turing machine can make beginning on a blank tape and stopping after a finite number of shifts. $\mathsf{R}(n)$ is the maximum number of distinct cells an n card Turing machine can scan beginning with a blank tape and stopping after a finite number of shifts.

Rado has stated that $\mathsf{S}(n) \leqslant (n+1)\Sigma(5n)2^{\Sigma(5n)}$. This is a result of the fact that $\mathsf{S}(n) \leqslant (n+1)\mathsf{R}(n)2^{\mathsf{R}(n)}$, since the number of shifts a machine makes

*I obtained this result while attending Prof. Hans Zassenhaus' Seminar in Experimental Number Theory at Ohio State University, Columbus, Ohio, during the Summer of 1966.

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