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A THEOREM ON n-TUPLES WHICH IS EQUIVALENT TO THE WELL-ORDERING THEOREM

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Using a form of the well-ordering theorem which is due to A. Levy [3] it is possible to generalize a result of B. Sobociński [6] and prove the following theorem: *For all natural numbers n and k such that* $n > 2$ *and* $1 < k < n$ *the following proposition is equivalent to the well-ordering theorem.*

$\mathsf{P}(n, k)$: *For each set x which is not finite there exists a family N of unordered n-tuples of elements of x such that each unordered k-tuple of elements of x is a subset of exactly one of the elements of N.*

W. Sierpiński [5] proved that the axiom of choice implies $\mathsf{P}(3, 2)$ and B. Sobociński [6] proved that $\mathsf{P}(3, 2)$ implies the axiom of choice. Moreover, unknown to us, W. Frascella has also been working on this problem. In [1] Frascella proved that for each natural number $n > 2$, $\mathsf{P}(n, n-1)$ is equivalent to the axiom of choice and in [2] he proved the main results of this paper. However, Frascella's proofs are considerably different from ours.

Theorem 1. The well-ordering theorem implies that for all natural numbers n and k such that $n > 2$ *and* $1 < k < n$, $P(n, k)$ *holds.*

Proof: Let x be any set which is not finite and let n and k be natural numbers satisfying the hypotheses. By the well-ordering theorem there is an initial ordinal number ω_α such that $x \approx \omega_\alpha$. Let K be the set of all unordered k-tuples of elements of x. Then, it is also true that $K \approx \omega_\alpha$. (For example, we may well-order K by a relation R defined as follows: if $u, v \in K$, $u R v \Longleftrightarrow [(\max u < \max v)$ or $(\max u = \max v = w$ and $\max (u \sim \{w\}) < \max (v \sim \{w\}))$ or . . . or $(\max u = \max v$ and $\max (u \sim \{w\} = \max (v \sim \{w\})$ and . . . and $\min u \leqslant \min v)]$.) Let $K = \{k_\beta : \beta < \omega_\alpha\}$. In a similar manner we can well-order the set T of all unordered n-tuples of elements of x, so we also have $T \approx \omega_\alpha$. Let $T = \{t_\beta : \beta < \omega_\alpha\}$.

Now, we shall construct a subset N of T which satisfies $\mathsf{P}(n,k)$. Let $T_0 = \phi$. Suppose $T_\gamma \subseteq T$ has the property that for all $\beta < \gamma < \omega_\alpha$, k_β is a subset of exactly one element of T_γ and for all β such that $\gamma \leqslant \beta < \omega_\alpha$, k_β is

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