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FORMAL NONASSOCIATIVE NUMBER THEORY

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1. Introduction. The *logarithmetic*, first studied by I.M.H. Etherington (see, for example, [2]), of a nonassociative algebra has been found to bear some resemblance to the arithmetic of natural numbers. In [1], Evans has characterized this logarithmetic (i.e. the arithmetic of the indices of powers of the general element in a nonassociative algebra) by a set of axioms analogous to Peano's axioms and calls the resulting system "non-associative number theory."

In this paper, we shall formalize these "Peano-like" axioms and develop some of the properties of nonassociative number theory as theorems of the formal theory. In the last section it will be shown that formal nonassociative number theory, **N**, is both essentially undecidable and incomplete. This is accomplished by showing that **N** contains an essentially undecidable subtheory.

Few of the proofs of the theorems of **N** have all of the steps given. However, with the metamathematical remarks given, it should be an easy matter for the interested reader to supply complete proofs.

2. An axiom system for nonassociative number theory. We define **N** (formal nonassociative number theory) to be the first-order theory whose only individual constant is a_1, whose only predicate letter is A_1^2, and whose only function letters are f_1^2, f_2^2, and f_3^2. We write 1 for a_1, $x_1 = x_2$ for $A_1^2(x_1, x_2)$, $x_1 + x_2$ for $f_1^2(x_1, x_2)$, $x_1 \cdot x_2$ for $f_2^2(x_1, x_2)$, *and* $x_1^{x_2}$ for $f_3^2(x_1, x_2)$. The proper axioms of **N** are the following:

(N1) $x_1 = x_2 \supset (x_1 = x_3 \supset x_2 = x_3)$
(N2) $x_1 = x_2 \supset (x_1 + x_3 = x_2 + x_3)$
(N3) $x_1 = x_2 \supset (x_3 + x_1 = x_3 + x_2)$
(N4) $x_1 + x_2 \neq 1$
(N5) $x_1 + x_2 = x_3 + x_4 \supset (x_1 = x_3 \wedge x_2 = x_4)$
(N6) $x_1 \cdot 1 = x_1$
(N7) $x_1 \cdot (x_2 + x_3) = x_1 \cdot x_2 + x_1 \cdot x_3$
(N8) $x_1^1 = x_1$
(N9) $x_1^{x_2 + x_3} = x_1^{x_2} \cdot x_1^{x_3}$
(N10) (Nonassociative Induction):

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