IMPLICATIONLESS WFFS IN IC

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Let Σ be the set of all wffs. of the Intuitionist Propositional Calculus (hereafter IC.) Let Σ_1, $\Sigma_1 \subset \Sigma$, be the set of wffs. which contain only the conjunction and negation signs. Similarly let Σ_2, $\Sigma_1 \subset \Sigma_2 \subset \Sigma$, be the set of wffs. which do not contain the implication sign. For wffs. $P_i \,\varepsilon\, \Sigma_1$ we have the well-known "representation theorem" of Gödel, [1], based on a result of Glivenko, that

$$\vdash_{\mathsf{IC}} P_i \ \ iff \ \ \vdash_{\mathsf{HA}} P_i$$

where HA. is the classical propositional calculus. An analogous representation theorem for Σ_2 can be shown to follow from a result of Jankov, [2]. We note firstly.

THEOREM 1 There is no finite characteristic model for Σ_2

Proof Consider the wff $A \equiv \bigvee\limits_{\substack{i < j \\ 2 \le j \le k}} \neg\, (a_i \wedge \neg\, a_j)$ and proceed exactly as in

Gödel's proof, *cf.* [1], that there exists no finite characteristic model for Σ.

LEMMA 1 Every wff $P_i \,\varepsilon\, \Sigma_2$ is equivalent to a wff $A_k \,\varepsilon\, \Sigma_2$ where A_k is of the form $\bigvee\limits_{1 \le i \le k} a_i$ and each $a_i \,\varepsilon\, \Sigma_1$.

Proof   By induction on the number of connectives in P_i using the equivalence $\neg\,(a \vee b) \equiv \neg\, a \wedge \neg\, b$ and the distributive laws.

LEMMA 2   For every wff $P_i \,\varepsilon\, \Sigma_2$, $\vdash_{\mathsf{IC}} P_i$ iff $A_k^{\le}$ vanishes identically in $\Gamma(\mathsf{B}^k)$ where A_k is the normal form of P_i as defined in the preceding lemma, $A_k^{\le}$ is the lattice polynomial (for lattice background, see [3]) corresponding to A_k and $\Gamma(\mathsf{B}^k)$ is the lattice obtained by applying the Jaśkowski operation Γ to the direct product of the 2-element Boolean lattice with itself k times.

Proof   If $\vdash_{\mathsf{IC}} P_i$ then $A_k^{\le}$ will vanish identically in $\Gamma(\mathsf{B}^k)$ since $\Gamma(\mathsf{B}^k)$ is a finite distributive lattice. For the converse, suppose P_i is not a theorem