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NECESSITY AND TICKET ENTAILMENT

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In [1], Anderson introduces the system $\mathbf{P}_I$, i.e. the implicational fragment of the system **P** of "ticket entailment," for which the following axiom schemas are given:

$\mathbf{P}_I1$. $A \rightarrow A$
$\mathbf{P}_I2$. $A \rightarrow B \rightarrow . B \rightarrow C \rightarrow . A \rightarrow C$
$\mathbf{P}_I3$. $A \rightarrow B \rightarrow . C \rightarrow A \rightarrow . C \rightarrow B$
$\mathbf{P}_I4$. $(A \rightarrow . A \rightarrow B) \rightarrow . A \rightarrow B$.

$\rightarrow$**E** (*modus ponens*) is the sole primitive inference rule of $\mathbf{P}_I$. A theory of necessity cannot be developed in $\mathbf{P}_I$ (as in $\mathbf{E}_I$, i.e. the implicational fragment of **E**) via the definition

$$\mathrm{N}A =_{df} A \rightarrow A \rightarrow A$$

since $A \rightarrow A \rightarrow A \rightarrow A$ (i.e. $\mathrm{N}A \rightarrow A$) is not provable in $\mathbf{P}_I$. In [2], the question is raised whether there is any function f of a single variable A definable in $\mathbf{P}_I$ which makes $f(A)$ look like "necessarily A," i.e. such that

(1) $\vdash f(A) \rightarrow A$
(2) $\dashv A \rightarrow f(A)$
(3) *if* $\vdash A$ *then* $\vdash f(A)$
(4) *if* $\vdash A \rightarrow B$ *then* $\vdash f(A) \rightarrow f(B)$.

In [3, §6], the question is raised again with slightly different conditions on f: (1)-(3) above, and

(5) $\vdash A \rightarrow B \rightarrow . f(A) \rightarrow f(B)$.

This last formulation of the question is answered by the following

Theorem. *There is no function f definable in* $\mathbf{P}_I$ *satisfying conditions* (1)-(3) *and* (5).

Proof. Assume on the contrary that there is such a function. Consider the matrix (with designated values 2 and 3)

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