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REAL FIELDS WITH CHARACTERIZATION OF THE NATURAL NUMBERS

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Introduction. This paper is a sequel to [3]. Its purpose is two-fold: On the one hand, it is to inform the reader of the state of affairs regarding the concept of structures being elementarily closed (relative to the natural numbers) since the publication of Theorems 3D, 4A of [2] where the concept (without name) was introduced for the first time. And, on the other hand, our purpose is to give some clarification to results of [3]. In [3] the term "elementarily closed" was introduced to apply to the general structure which satisfies the conclusions of Theorems 3D, 4A of [2]. However, recent studies have led us to conclude that a stronger form of our definition is more interesting and natural from the point of view of our results in [2]. We shall attempt to be more specific after giving a precise frame of reference.

1 *Basic Definitions and Remarks.* We are here interested in structures of the form $\mathcal{F} = \{F, \mathcal{N}_0, +, \cdot, \leq, 0\}$, where F is the set-part for a field of characteristic zero, $\mathcal{N}_0$ the set of natural numbers, "+" and "·" the ternary relations of addition and multiplication, respectively, and "≤" the binary relation of order. (In some cases we choose to drop "<".) By the language of $\mathcal{F}$, say $\Sigma_{\mathcal{F}}$, is meant a convenient formulation of the lower predicate calculus which contains the extralogical constants $\mathsf{N}(x)$, $\mathsf{E}(x, y)$, $\mathsf{S}(x, y, z)$, $\mathsf{P}(x, y, z)$ and $\mathsf{H}(x, y)$ whose intended interpretations are "$x \epsilon \mathcal{N}_0$", $x = y$, $x + y = z$, $x \cdot y = z$, and $x < y$, respectively.

Let $\mathcal{F}$ and $\mathcal{G}$ be two structures and $A \subseteq F \cup G$. Recall that $\mathcal{F}$ and $\mathcal{G}$ are said to be *elementarily equivalent* with respect to A in case $\mathcal{F} \models X$ if and only if $\mathcal{G} \models X$, for any sentence X which is defined in the language of $\mathcal{F}$ (and $\mathcal{G}$) and whose individual constants correspond to elements of A. $\mathcal{F}$ and $\mathcal{G}$ are said to be *elementarily equivalent* in case they are elementarily equivalent with respect to ϕ. $\mathcal{F}$ is an *elementary extension* of $\mathcal{G}$ in case $\mathcal{F}$ is an extension of $\mathcal{G}$ and $\mathcal{F}$ and $\mathcal{G}$ are elementarily equivalent with respect to G.

It is interesting to observe that a certain abnormality occurs for

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