Notre Dame Journal of Formal Logic
Volume XIII, Number 2, April 1972
NDJFAM

INDUCTION ON FIELDS OF BINARY RELATIONS

W. RUSSELL BELDING

In [1] the following principle of induction, introduced by Montague in [2],

A. *If* $\varphi(x)$ *is a formula not containing the variable* y *and* R *is a well-founded relation, then*

$$(y)(y \in \mathsf{Fld}R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in \mathsf{Fld}R \rightarrow \varphi(y))$$

is proved in the field of G.B. set theory. It is shown below that the restriction that R be well-founded can be removed and the induction will still hold provided a restriction is placed on the formula $\varphi(x)$. The relationship between the various induction principles of [1] and the induction principle proved in this paper (Theorem 1) is discussed. The notation and definitions used in this paper are explained and defined in [1]. The relations considered in this paper are always binary relations.

The following theorem gives a new sufficient condition for induction of binary relations.

Theorem 1. (Induction Principle **E**). *For every* R *and every* $\varphi(x)$, *if* R *is a binary relation,* $\varphi(x)$ *a formula not containing the variable* y *and* $\varphi(x)$ *has the property that for every sequence of sets* $\{a_n\}_{n<\omega}$ *such that* $a_{n+1}Ra_n$ *for every* $n \geqslant 0$, *there is at least one integer* $m \geqslant 0$ *such that* $\varphi(a_m)$ *holds, then*

$$(y)(y \in \mathsf{Fld}R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in \mathsf{Fld}R \rightarrow \varphi(y)).$$

Proof: An indirect proof is used. Assume the hypothesis and suppose the induction fails. That is,

(1) $(y)(y \in \mathsf{Fld}R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y))$
(2) $(\exists y)(y \in \mathsf{Fld}R \wedge \sim \varphi(y))$

Suppose a_0 is such that $a_0 \in \mathsf{Fld}R$ and $\sim \varphi(a_0)$. First suppose that a_0 has no predecessor or that φ holds for every predecessor of a_0. Then clearly

(3) $a_0 \in \mathsf{Fld}R \wedge (x)(xRa_0 \rightarrow \varphi(x))$

By (3) and (1) it follows that $\varphi(a_0)$ holds, contrary to (2). Thus,

Received August 4, 1970