Notre Dame Journal of Formal Logic
Volume XIII, Number 2, April 1972
NDJFAM

TRANSITIVITY, SUPERTRANSITIVITY AND INDUCTION

W. RUSSELL BELDING, RICHARD L. POSS and
PAUL J. WELSH, Jr.

In [3] and [6] Montague and Tarski, respectively, published without proofs the following general principles of induction as theorems of Zermelo-Fraenkel set theory (ZF).

A. (Montague) *If the formula* $\varphi(x)$ *does not contain the variable* y *and the relation* R *is well-founded then*

$$(y)(y \in \mathsf{Fld}R \wedge (x)(xRy \rightarrow \varphi(x)) \rightarrow \varphi(y)) \rightarrow (y)(y \in \mathsf{Fld}R \rightarrow \varphi(y)).$$

B. (Tarski) *If the formula* $\varphi(x)$ *does not contain the variable* y *then*

$$(y)[(x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(\varphi(y)).$$

In this paper we shall present, in Gödel-Bernays set theory (GB), some results concerning general principles of induction, relating them to **A** and **B** above. In section **1** we list our notation and definitions. In section **2**, since Montague and Tarski published their results without proofs, we shall for the sake of completeness give our proofs of their results. We shall also prove the following general induction principle:

C. *If the formula* $\varphi(x)$ *does not contain the variable* y *and* A *is a transitive class then*

$$(y)[y \in A \wedge (x)(x \in y \rightarrow \varphi(x)) \rightarrow \varphi(y)] \rightarrow (y)(y \in A \rightarrow \varphi(y)).$$

We also show that it is necessary that A be transitive for this principle to hold. We also present a transitive decomposition formula for classes. In section **3** we introduce the notion of supertransitivity for classes, give examples of supertransitive classes and discuss the relationship between supertransitivity and transitivity. Finally in section **4** we present a supertransitive decomposition formula for classes and prove the following induction principle for supertransitive classes.

D. *Let* A *be a supertransitive class and* $\varphi(x)$ *be a formula not containing the variable* y*. If* φ *has the property that for every sequence of sets* $\{a_n\}_{n<\omega}$

Received July 1, 1970