Notre Dame Journal of Formal Logic
Volume XIII, Number 1, January 1972
NDJFAM

ON THE RECURSIVE UNSOLVABILITY OF THE PROVABILITY OF THE DEDUCTION THEOREM IN PARTIAL PROPOSITIONAL CALCULI

D. BOLLMAN and M. TAPIA

1. *Introduction.** In a recent paper, Pogorzelski [2] proved the existence of a "weakest" partial propositional (implicational) calculus. Such a calculus is the weakest in the sense that its rules and axioms are among the derived rules and theorems respectively of any partial propositional calculus in which the deduction theorem holds. This result suggests the following question: Is it possible to algorithmically determine of an arbitrarily given partial propositional calculus whether or not the deduction theorem holds? In this paper we prove that the answer to this question is negative.

To accomplish our goal, we consider a special subclass of partial propositional calculi, namely those in which the rules of inference consist of modus ponens and substitution. It is known that the decision problem for the latter calculi is recursively unsolvable and we shall use Singletary's construction of such calculi in [3]. The proof that the problem of determining whether or not the deduction theorem holds for such a calculus arbitrarily given will parallel Yntema's proof of the recursive unsolvability of the completeness problem for a more restricted class of partial propositional calculi in [4]. Finally, using a well-known result of Boone [1], it will be an easy matter to show that for every recursively enumerable degree of unsolvability D, there exists a class of partial propositional calculi such that the problem of determining whether or not the deduction theorem holds for any member of the class is of degree D.

2. *Preliminaries.* For the purposes of this paper, we shall define a *generalized partial propositional calculus* to be a formal system whose

*This paper is part of a thesis written under the direction of Professor D. Bollman and submitted to the University of Puerto Rico at Mayagüez in July, 1970 in partial fulfillment of the requirements for the degree of Master of Science.

Received June 13, 1970