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WEIERSTRASS'S FINAL THEOREM OF ARITHMETIC IS NOT FINAL

F. G. ASENJO and J. M. McKEAN

Weierstrass proved that the system of complex numbers $\mathcal{C}$ is the first and last algebraic extension of the field of real numbers $\mathcal{R}$ which has the properties that (i) it is itself a field, that (ii) it is algebraically closed, and that (iii) it is a finite dimensional vector space over $\mathcal{R}$. As a consequence, the so-called "final theorem of arithmetic" follows:

It is impossible to extend the number system beyond $\mathcal{C}$ into structures that preserve all the formal laws of arithmetic (most significantly, the field-theoretic ones)—that is, full-fledged arithmetic ends with $\mathcal{C}$.

This theorem is principally based on the fact that $\mathcal{C}$ is a finite dimensional vector space over $\mathcal{R}$; however, this puts too much reliance on a condition extraneous to both the program and the spirit of Hankel's classical formal laws of arithmetic. Since $\mathcal{C}$ is both algebraically closed and a vector space over $\mathcal{R}$, one tends to lose sight of the obvious fact that property (iii) is not a consequence of (i).

This problem arises: Is it possible to generalize the final theorem of arithmetic to fields obtained by adjoining the root of $x^2 + 1 = 0$ to any field $\mathcal{R}^*$ that contains $\mathcal{R}$ and is obtained from $\mathcal{R}$ by strictly finitistic procedures (i.e., finitistic rules of formation, quotient structures, and the like) rather than by ultraproducts or some other infinitistic device? The answer to this question is no, as we shall outline. To be specific, an extension $\mathcal{R}^*$ of $\mathcal{R}$ can be finitistically constructed in which $x^2 + 1 = 0$ is not solvable, and in which the field $\mathcal{C}^*$ obtained by adjoining i to $\mathcal{R}^*$ is not algebraically closed, and such that from $\mathcal{R}^*$ an unlimited number of algebraic field extensions can be obtained. These extensions yield an infinite sequence of number systems, all of which are fields and none of which is algebraically closed. Thus, the final theorem of arithmetic fails for $\mathcal{C}^*$ as well as for all its algebraic extensions. It is possible to expand the concept of number from $\mathcal{R}^*$ indefinitely.

In [1] a number system $\mathcal{N}^*$ was introduced that is an extension of the system of natural numbers $\mathcal{N}$. Members of $\mathcal{N}^*$—term-relation numbers—are obtained as by-products of formalizing some of the properties of

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