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A CLASS OF MODELS FOR INTERMEDIATE LOGICS

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Let α be an ordinal, $c(\alpha)$ its cardinality and B a $c(\alpha)$-field of sets, with union $+$, intersection $\cdot$ and complementation $'$. By $\mathsf{L}_\alpha(B)$ we denote the set of weakly decreasing functions from α into B. A lattice structure is defined on $\mathsf{L}_\alpha(B)$ by putting

$$(f+g)(\kappa) = f(\kappa) + g(\kappa)$$
$$(f \cdot g)(\kappa) = f(\kappa) \cdot g(\kappa)$$

for all $\kappa \leq \alpha$ and $f, g \in \mathsf{L}_\alpha(B)$. There is a zero 0 in $\mathsf{L}_\alpha(B)$ and a one 1. As is well-known $\mathsf{L}_\alpha(B)$ is not complemented for $\alpha > 1$. However a relatively pseudocomplemented structure can be defined on $\mathsf{L}_\alpha(B)$.

Definition: For $f, g \in \mathsf{L}_\alpha(B)$ let $f \to g$ be defined by

$$(f \to g)(\kappa) = \sum_{\rho \leq \kappa} f(\rho)' \cdot \prod_{\sigma < \rho} g(\sigma) + g(\kappa)$$

Remarks:

1. The void product $\prod_{\sigma<1} g(\sigma)$ is put equal to $1 \in B$.
2. Notice the following recursive relation

$$(f \to g)(\kappa+1) = f(\kappa+1)' \cdot (f \to g)(\kappa) + g(\kappa+1) .$$

Theorem 1: *If $f, g \in \mathsf{L}_\alpha(B)$, then $f \to g \in \mathsf{L}_\alpha(B)$.*

Proof. By the assumed nature of B the $(f \to g)(\kappa)$ are in B for all $\kappa \leq \alpha$. If $\tau < \kappa$ then

$$\begin{aligned}(f \to g)(\kappa) &= \sum_{\rho \leq \kappa} f(\rho)' \cdot \prod_{\sigma<\rho} g(\sigma) + g(\kappa)\\ &= \sum_{\rho \leq \tau} f(\rho)' \cdot \prod_{\sigma<\rho} g(\sigma) + \sum_{\tau < \rho \leq \kappa} f(\rho)' \cdot \prod_{\sigma<\rho} g(\sigma) + g(\kappa)\\ &\leq \sum_{\rho \leq \tau} f(\rho)' \cdot \prod_{\sigma<\rho} g(\sigma) + g(\tau)\\ &= (f \to g)(\tau)\end{aligned}$$

i.e. $f \to g$ is weakly decreasing.

Theorem 2: *$\langle \mathsf{L}_\alpha(B), +, \cdot, \to \rangle$ is relatively pseudocomplemented.*

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