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ON THE INTERPRETABILITY OF ARITHMETIC IN SET THEORY

GEORGE E. COLLINS and J. D. HALPERN

In 1950, Wanda Szmielew and Alfred Tarski [1] announced that the theory **Q**, a finitely axiomatizable essentially undecidable fragment of arithmetic, is interpretable in a small fragment **S** of set theory. The fragment **S** is so small that it is easily interpretable in any of the known formalizations of class or set theory with or without urelements and remains so interpretable even if all axioms of infinity are removed (most other axioms can be deleted also.) Furthermore, **S** is finitely axiomatized, it has three axioms, and even though its non-logical constants consist of one unary and one binary predicate symbol, the modification resulting from simple deletion of the unary symbol gives a stronger theory and hence gives another proof that first order predicate logic with a binary predicate symbol is undecidable, as is remarked in [2] (p. 34).

In 1964, the first author became interested in the result and no proof being available in the literature, the two of us devised a proof of it, an outline of which we communicated to Professor Tarski. Subsequently, Professor Tarski encouraged us to publish the proof which we do herewith.*

The proof we give appears to have some value beyond establishing the interpretability of **Q** in **S**. For instance one can prove from the definition of + in **S** that $0 + \{\{1\}\} \neq \{\{1\}\} + 0$; hence the commutative law for addition is not provable in **Q**. This raises a question, alien to the original motivation but we believe interesting in a technical sense. Can one interpret the theory **Q**, enriched by the addition of some or all of the commutative, associative and distributive laws, in the theory **S**?

The theories **Q** and **S** are the first order theories whose axioms are as follows, ([2] pp. 51 and 34):

Theory **Q**:

Q1. $Sx = Sy \rightarrow x = y$

Q2. $0 \neq Sy$

Q3. $x \neq 0 \rightarrow (\exists y)(x = Sy)$

Q4. $x + 0 = x$

Q5. $x + Sy = S(x + y)$

Q6. $x \cdot 0 = 0$

Q7. $x \cdot Sy = (x \cdot y) + x$

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