Notre Dame Journal of Formal Logic
Volume XI, Number 4, October 1970

A PARADOX IN ILLATIVE COMBINATORY LOGIC

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Curry, in [1] and [2], has shown the inconsistency of a system of illative combinatory logic containing the axiom:

$$\vdash \mathsf{H}^k\mathfrak{X} \text{ for all obs } \mathfrak{X},$$

for $k = 2$ (and 1). ("$\mathsf{H}\mathfrak{X}$" stands for "$\mathfrak{X}$ is a proposition".) He also stated that the inconsistency held for $k > 2$, this more general result is proved below. Assume the following:

A1	$\mathsf{H}\mathfrak{X}, \mathsf{H}\mathfrak{Y}, \mathsf{H}\mathfrak{Z} \vdash \mathfrak{X} \supset . \mathfrak{Y} \supset \mathfrak{Z} : \supset : \mathfrak{X} \supset \mathfrak{Y} . \supset . \mathfrak{X} \supset \mathfrak{Z}$.
A2	$\mathsf{H}\mathfrak{X}, \mathsf{H}\mathfrak{Y} \vdash \mathfrak{X} \supset . \mathfrak{Y} \supset \mathfrak{X}$.
A3	$\mathfrak{X}, \mathsf{P}\mathfrak{X}\mathfrak{Y} \vdash \mathfrak{Y}$.
A4	$\mathfrak{X} \vdash \mathsf{H}\mathfrak{X}$.
A5	$\vdash \mathsf{H}^{k+1}\mathfrak{X}$ *for any* $\mathfrak{X}$ *and* $k \geqslant 0$.
A6	$\vdash \mathsf{H}\mathfrak{A}$.
A7	*If* $\vdash \mathsf{H}\mathfrak{X}$ *and* $\mathfrak{X} \vdash \mathsf{H}\mathfrak{Y}$ *then* $\vdash \mathsf{H}(\mathsf{P}\mathfrak{X}\mathfrak{Y})$.

From *A1*, *A2*, *A3* and *A7* it follows (as is proved in [4]) that if $\mathsf{T}(\mathfrak{X}_1, \ldots, \mathfrak{X}_n)$ is any theorem of pure implicational intuitionistic propositional calculus for indeterminates $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$, then

$$\mathsf{H}\mathfrak{X}_1, \mathsf{H}\mathfrak{X}_2, \ldots, \mathsf{H}\mathfrak{X}_n \vdash \mathsf{T}(\mathfrak{X}_1, \ldots, \mathfrak{X}_n).$$

This fact is used in several places below.

Let $$G_0 \equiv [x] . x \supset \mathfrak{A},$$

and for $n \geqslant 0$ let

$$G_{n+1} \equiv [x] . \mathsf{H}^{n+1}x \supset G_n x.$$

Now

$$\mathsf{H}^{n+1}x \vdash \mathsf{H}(G_n x) \tag{1}$$

is proved by induction, thus:

By *A6* and *A7*

$$\mathsf{H}x \vdash \mathsf{H}(G_0 x).$$

Received November 29, 1969