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BINARY CLOSURE-ALGEBRAIC OPERATIONS THAT ARE FUNCTIONALLY COMPLETE

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1. *Preliminaries*.* It is well known that the modal system S4 is related to closure algebras in the same way that the classical propositional calculus is related to Boolean algebras, namely: a wff is a theorem of S4 if and only if its algebraic transliteration is valid in every closure algebra ([3], p. 130). Consequently, many results about closure algebras carry over to S4, and conversely. In this paper we exploit the aforementioned relationship to introduce binary closure-algebraic operations that are functionally complete in closure algebras in the same sense that the operations of nonunion and nonintersection are functionally complete in Boolean algebras. By a *closure-algebraic operation of* a closure algebra $\langle K, -, \cap, *\rangle$ we shall understand an operation on K that is generable by finite composition from the operations $*$ (closure), $\cap$ (intersection), and $-$ (complementation). A set $\triangle$ of closure-algebraic operations of a closure algebra $\langle K, -, \cap, *\rangle$ shall be called *functionally complete in* $\langle K, -, \cap, *\rangle$ if every closure-algebraic operation of $\langle K, -, \cap, *\rangle$ can be generated by finite composition from the members of $\triangle$. We can now state precisely the theorem that will be proved:

If $\langle K, -, \cap, *\rangle$ *is a closure algebra, then* (*the unit set of*) *the binary closure-algebraic operation* $\ast$ *of* $\langle K, -, \cap, *\rangle$ *is functionally complete in* $\langle K, -, \cap, *\rangle$, *where*

$$A \ast B =_{Df} [-(-A \cap *A \cap -*B) \cup A] \cap [(-A \cap *A \cap -*B) \cup -(A \cap B)].$$

The same is also true of the closure-algebraic operation dual to $\ast$.

2. *Proof of Theorem*. In view of the aforementioned relationship between S4 and closure algebras, it is sufficient proof of the theorem to show that the binary connective '$\ast$' serves by itself to define the S4 connectives '$\sim$', '$\cdot$', and '$\Diamond$' (or '$\Box$'), where

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