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EXPRESSIBILITY IN TYPE THEORY

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§1. *Introduction*

This paper is based on a conception of mathematics not as a system of statements about "mathematical objects" but as a system of derived rules of inference which may be applied to physical objects. It aims to build a foundation for mathematics based on elementary rules of logic which is independent, so far as possible, of ontological presuppositions. The concept of expressibility is introduced here mainly for use in constructing such a foundation. It is a generalization and adaptation to type theory of the "strong Γ-consistency" defined by Henkin in [4].

Let $\mathfrak{L}$ be a simple theory of types. Let Γ be a set of individual or predicate constants in $\mathfrak{L}$ such that all members of Γ belong to the same type. Let γ be a predicate in $\mathfrak{L}$ such that $\gamma(\mathbf{a})$ is wf if $\mathbf{a} \epsilon \Gamma$ (where $\mathbf{a}$ is used autonymously). We shall say that Γ is *expressible by* γ in $\mathfrak{L}$ if there exists a complete and consistent extension $\mathfrak{M}$ of $\mathfrak{L}$ such that, for every constant $\mathbf{b}$ of the relevant type, $\gamma(\mathbf{b})$ is valid in $\mathfrak{M}$ iff, for some $\mathbf{a} \epsilon \Gamma$, the wff $\mathbf{b} = \mathbf{a}$ is valid in $\mathfrak{M}$.

It will be shown that a simple theory of types can serve as a satisfactory foundation for mathematics if and only if certain meta-predicates (i.e. predicates defined in the metatheory) are expressible by predicates in the system. Incidentally, we shall find that certain important problems of consistency, e.g. ω-consistency of number theory, consistency of choice axioms, are reducible to problems of expressibility.

Let us say that a system $\mathfrak{L}$ of type theory is *adequate for mathematics* iff

(1) the set of wffs of $\mathfrak{L}$ is recursive, its set of theorems recursively enumerable,

(2) there is a designated class of individual constants called *names* (for objects) such that

(a) if $\mathbf{a}$, $\mathbf{b}$ are typographically distinct names, then $\vdash \mathbf{a} \neq \mathbf{b}$ in $\mathfrak{L}$,

(b) if $A(x_1, \ldots, x_n)$ is a wff in which the individual variables $x_1, \ldots, x_n$ occur free such that

$$\vdash \exists x_1 \ldots \exists x_n A(x_1, \ldots, x_n) \text{ in } \mathfrak{L},$$

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