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A SEMI-COMPLETENESS THEOREM*

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In [2] and [3] the hyperprojective hierarchy of subsets of $\mathfrak{N}$ (the functions from **N**, the natural numbers, into **N**) was constructed using the hyperarithmetic hierarchy as a model. As a consequence, many properties of the hyperarithmetic sets have perfect analogues in the hyperprojective sets. In this paper we consider the problem of finding a "projective analogue" for the following important fact connected with the hyperarithmetic hierarchy:

The set of Gödel numbers of the recursive well-orderings is a complete π_1^1 set of natural numbers (see [5]).

In [2] it was shown that the set of indices of the projective well-orderings[1] of subsets of $\mathfrak{N}$ is a Δ_1^2 set[2] and thus cannot be a "complete π_1^2 set" in any natural sense, as our analogy would have it. The difficulty is that in order to express the notion "there is no countable descending chain of functions such that . . ." one needs only a function quantifier, not a quantifier over functions from $\mathfrak{N}$ into **N**. Thus we are led to consider the collection W^* of indices of those projective linear orderings having no *uncountable* descending chains.

In this paper we will show that W^* has a semi-completeness property with respect to a subclass of the π_1^2 sets. Our proof will assume the existence of a projective well-ordering $<^*$ of all of $\mathfrak{N}$ such that $\mathfrak{N}$ in this ordering is order-isomorphic to the first uncountable ordinal Ω. This assumption is consistent with the usual axioms for set theory, since the existence of a Δ_2^1 well-ordering of $\mathfrak{N}$ (of length Ω) follows from the Axiom of Constructibility [1].

Definition. If a subset B of $\mathfrak{N}$ is linearly ordered by some relation $<B$,

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