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THE APPLICATION OF TERNARY SEMIGROUPS TO THE STUDY OF n-VALUED SHEFFER FUNCTIONS

JAMES ROSENBERG

If we consider a set of one-variable truth functions we can define on this set a product operation, namely composition. If we assume that this set is closed under composition then the algebraic structure which results is that of a semigroup. In this paper we extend this notion to consider sets of binary truth functions by introducing the concept of a ternary semigroup, and prove a theorem concerning n-valued Sheffer functions. (For one of the most recent papers on this subject with an excellent bibliography see [1].) The methods presented are entirely algebraic, but then it may be argued that problems involving the characterization of n-valued Sheffer functions belong more properly to abstract algebra than symbolic logic.

1. *Definition:* A *ternary semigroup* is a set G with a closed ternary product operation fgh such that for any $f, g, h, x, y \varepsilon G$,

$$(fgh)xy = f(gxy)\ (hxy)^1$$

For the best example of a ternary semigroup consider a set F of binary functions on a set T—i.e. a set of functions which map $T \times T \rightarrow T$. Define a ternary product on F by the *ternary composition map*:

$$fgh(x,y) = f(g(x,y), h(x,y))\ x, y\ \varepsilon\ T,\ f, g, h\ \varepsilon\ F$$

The similarity between this definition and the condition of Definition 1 will readily be seen. In fact, if we assume that for $f,g,h\, \varepsilon\, F\ fgh\, \varepsilon\, F$ then F is a ternary semigroup under composition. In this case F will be said to be a ternary semigroup *acting on T*. We will define isomorphism in the natural way, namely two ternary semigroups G and H will be said to be *isomorphic* iff there exists a 1-1 onto map $\phi: G \rightarrow H$ such that $\phi(abc) = \phi(a)\phi(b)\phi(c)$ for any $a, b, c\, \varepsilon\, G$.

2. *Theorem: For any ternary semigroup G there is a set T and a ternary semigroup H acting on T such that G is isomorphic to H.*

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1. If a system of notation were used in which function arguments were placed on the *left* of the function symbol the condition would be written $xy(fgh) = (xyf)(xyg)h$.

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