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A FORMALISATION OF THE ARITHMETIC OF THE ORDINALS LESS THAN ω^ω

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Some of the results of ordinal arithmetic can be derived from a multi-successor equation calculus. The initial functions are:

(i) the zero function $\mathsf{N}(x) = 0$
(ii) the identity function $\mathsf{I}(x) = x$.

These two functions are implicit. In addition there are:

(iii) a countable number of successor functions $\mathsf{S}_0, \mathsf{S}_1, \mathsf{S}_2, \ldots$.

The successor functions are restricted by the axioms

A $$\mathsf{S}_\mu \mathsf{S}_\nu = \mathsf{S}_\mu \text{ if } \mu > \nu$$
B $$\mathsf{S}_a \mathsf{S}_b \ldots \mathsf{S}_q = \mathsf{S}_{a'} \mathsf{S}_{b'} \ldots \mathsf{S}_{q'}$$

with $a \leq b \leq \ldots \leq q$ and $a' \leq b' \leq \ldots \leq q'$ if and only if $a = a'$, $b = b'$, $\ldots q = q'$.

A function may be defined explicitly, or by recursion in the following way

$$F(x, 0) = a(x)$$
$$F(x, \mathsf{S}_\mu y) = b_\mu(x, y, F(x, y))$$

from previously defined functions $a(x)$ and $b_\mu(x, y, z)$ (for all μ) if the b_μ obey the following identity imposed by **A**:

C $$b_\mu(x, \mathsf{S}_\nu y, b_\nu(x, y, z)) = b_\mu(x, y, z) \text{ if } \nu < \mu,$$

The rules of inference are the following schemata

Sb$_1$ $$\frac{F(x) = G(x)}{F(A) = G(A)}$$

Sb$_2$ $$\frac{A = B}{F(A) = F(B)}$$

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