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ON KLEENE'S RECURSIVE REALIZABILITY AS AN INTERPRETATION FOR INTUITIONISTIC ELEMENTARY NUMBER THEORY

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Kleene (*Introduction to Metamathematics*, p. 501 ff.) has shown that when intuitionistic elementary number theory is interpreted in terms of recursive realizability certain elementary number theoretic statements are classically true but intuitionistically unacceptable; and that their negations are classically false but intuitionistically acceptable. Examples of such statements are (for a suitably chosen predicate A(x)): 1) excluded middle; 2) the least number principle; 3) the double negation and universal closure of (1) and (2). I shall show that a statement classically equivalent to the induction axiom has this same property, and why this is so. I shall then argue that this interpretation of intuitionistic number theory is fundamentally incorrect. And finally I shall suggest another interpretation that renders (1), (2) and (3) intuitionistically acceptable for that predicate A(x).

PART I

The formal system (**Z**) for intuitionistic elementary number theory (**I.M.**, p. 82) differs from the classical (**T**) in just one axiom:

$$\neg\neg A \supset A \qquad \text{(classical)}$$
$$\neg A \supset (A \supset B) \quad \text{(intuitionistic)}$$

The induction axiom in both (**Z**) and (**T**) is:

$$(1) \quad (A(0) \,\&\, (x)(A(x) \supset A(x'))) \supset A(x)$$

The interpretation as recursive realizability proceeds as follows: (x is a variable; x is a natural number; **x** is the formal numeral corresponding to x.)

(A) 1. The number e realizes a closed prime formula P (one without free variables and logical symbols) if $e = 0$ and P is recursively true.
If A and B are any closed formulas (without free variables):

2) e realizes A & B if $e = 2^a \cdot 3^b$ where a realizes A and b realizes B.

3) e realizes A $\vee$ B if $e = 2^0 \cdot 3^a$ where a realizes A, or $e = 2^1 \cdot 3^b$ where b realizes B.

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