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A HENKIN-STYLE COMPLETENESS PROOF FOR THE PURE IMPLICATIONAL CALCULUS

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Pollock has shown in [1] that Henkin-style completeness proofs can be obtained for deductive theories lacking negation, provided that disjunction is available. In this note, I show how to construct such proofs for implicational calculi without recourse to the special properties of disjunction exploited by Pollock. I shall run the argument through only for $\mathbf{PC_I}$, the pure implicational calculus, but the proof is easily adapted for richer theories as well.

For the sake of definiteness, we suppose $\mathbf{PC_I}$ to have

A1. $A \supset (B \supset A)$
A2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
A3. $((A \supset B) \supset A) \supset A$

as axiom-schemes and *modus ponens* as its only rule of inference. The relation '$\vdash$' of deducibility for $\mathbf{PC_I}$ is defined in the usual fashion.

Definition 1 A set Γ of formulas is *consistent* if $\Gamma \nvdash A$ for some formula A.

Definition 2 A set Γ of formulas is *maximal consistent* if

(1) Γ is consistent,
(2) $\Gamma \cup \{A\}$ is consistent, then $A \in \Gamma$.

We can now establish a familiar batch of theorems, the proofs of the first seven being straightforward and left to the reader.

Theorem 1 *If $A \in \Gamma$ or A is an axiom, then $\Gamma \vdash A$.*

Theorem 2 *If $\Gamma \vdash A \supset B$ and $\Gamma \vdash A$, then $\Gamma \vdash B$.*

Theorem 3 *If $\Gamma \cup \{A\} \vdash B$, then $\Gamma \vdash A \supset B$.*

Proof: As usual, using A1 and A2.

Theorem 4 *If $\Gamma \vdash A$, then $\Gamma \cup \Delta \vdash A$.*

Theorem 5 *If $\Gamma \vdash A$, then $\Delta \vdash A$ for some finite subset Δ of Γ.*

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