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# A THEOREM ON RECURSIVELY ENUMERABLE VECTOR SPACES

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This paper* is based on [1] and [2], but since we study only r.e. spaces, we prefer an exposition which is almost self-contained. Let $F$ be a countable field for which there is a one-to-one mapping $\phi$ from $F$ onto a recursive subset of $\varepsilon = (0, 1, \ldots)$ such that: $\phi(0_F) = 0$, $\phi(1_F) = 1$, $+_F$ and $\cdot_F$ correspond to partial recursive functions, $\phi(F) = (0, \ldots, q - 1)$, if $\mathrm{card}(F) = q$ and $\phi(F) = \varepsilon$, if $\mathrm{card}(F) = \aleph_0$. We write $\mathcal{W}_F$ for the vector space over $F$ which consists of all sequences of field elements with at most finitely many nonzero components, together with component-wise addition and scalar multiplication. Put

$$(1) \qquad \Phi\{x_n\} = \prod_{n \leqslant k} p_n{}^{\phi(x_n)} - 1, \text{ for } \{x_n\} \in \mathcal{W}_F,$$

where $p_0 = 2$, $p_n =$ the $n$'th odd prime, $k$ any number such that $x_n = 0_F$, for $n > k$. Then $\Phi$ maps $\mathcal{W}_F$ onto a vector space $\overline{U}_F = [\varepsilon_F, +, \cdot]$, where $\varepsilon_F$ is an infinite recursive set and $+$ and $\cdot$ are partial recursive functions. Note that the ordinary number 0 is also the zero element of $\overline{U}_F$. Set $\underline{e_n} = p_n - 1$, $\eta = (e_0, e_1, \ldots)$, then $\eta$ is an infinite recursive basis of $\overline{U}_F$, hence $\dim(\overline{U}_F) = \aleph_0$. The word "space" will be used in the sense of "subspace of $\overline{U}_F$". A space $\overline{V} = [\alpha, +, \cdot]$ is called *r.e.*, if the set $\alpha$ is r.e., *recursive*, if $\overline{V}$ is r.e. and has at least one r.e. complementary space, *decidable*, if $\alpha$ is a recursive set, i.e., if both $\alpha$ and $\varepsilon_F - \alpha$ are r.e.

The *purpose* of this paper is to examine the relationship between (I) $\overline{V}$ is a recursive space, and (II) $\overline{V}$ is a decidable space. We shall prove:

(a) *if $F$ is finite*, (I) $\Leftrightarrow$ (II),
(b) *if $F$ is infinite*, (I) $\Rightarrow$ (II), *but not conversely*.

A linearly independent subset of $\varepsilon_F$ is called a *repère*. According to [1], p. 2, there is an effective procedure which enables us to decide for any

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