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A SHORT POSTULATE-SYSTEM FOR ORTHOLATTICES

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By definition, *cf.*, [1], p. 52, an ortholattice is a lattice with universal bounds and a unary operation $^{\perp}$ satisfying:

$L1 \quad [a]: a \epsilon A . \supset . a \cap a^{\perp} = 0$
$L2 \quad [a]: a \epsilon A . \supset . a \cup a^{\perp} = 1$
$L3 \quad [ab]: a, b \epsilon A . \supset . (a \cup b)^{\perp} = a^{\perp} \cap b^{\perp}$
$L4 \quad [ab]: a, b \epsilon A . \supset . (a \cap b)^{\perp} = a^{\perp} \cup b^{\perp}$
$L5 \quad [a]: a \epsilon A . \supset . a = (a^{\perp})^{\perp}$

In this note it will be proved that:

(A) *Any algebraic system*

$$\mathfrak{A} = \langle A, \cup, \cap, ^{\perp} \rangle$$

where $\cup$ *and* $\cap$ *are two binary operations and* $^{\perp}$ *is a unary operation defined on the carrier set A, is an ortholattice, if it satisfies the following four mutually independent postulates:*

$B1 \quad [ab]: a, b \epsilon A . \supset . a \cup b = b \cup a$
$B2 \quad [ab]: a, b \epsilon A . \supset . a = a \cap (a \cup b)$
$B3 \quad [ab]: a, b \epsilon A . \supset . a = a \cup (b \cap b^{\perp})$
$B4 \quad [abc]: a, b, c \epsilon A . \supset . (a \cup b) \cup c = ((c^{\perp} \cap b^{\perp}) \cap a^{\perp})^{\perp}$ [1]

Proof:

1 Since it is self-evident that formulas *B1-B4* hold in the field of any ortholattice, only a converse should be proved. Hence, let us assume *B1-B4*. Then:

$B5 \quad [ab]: a, b \epsilon A . \supset . a \cap a^{\perp} = b \cap b^{\perp}$
$[B3, a/a \cap a^{\perp}; B1, a/a \cap a^{\perp}, b/b \cap b^{\perp}; B3, a/b \cap b^{\perp}, b/a]$
$D1 \quad [a]: a \epsilon A . \supset . a \cap a^{\perp} = 0 \qquad [B5]$
$B6 \quad [a]: a \epsilon A . \supset . a = a \cup 0 \qquad [B3; D1, a/b]$
$B7 \quad [a]: a \epsilon A . \supset . a = a \cap a \qquad [B2, b/0; B6]$
$B8 \quad [a]: a \epsilon A . \supset . a = 0 \cup a \qquad [B6; B1, b/0]$

1. Of course, in this postulate-system, the operations $\cup$, $\cap$, and $^{\perp}$ are not mutually independent.

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