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FORMULAS WITH TWO GENERALIZED QUANTIFIERS

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In this paper we give a partial solution to the two problems Yasuhara presents at the end of [2]. Yasuhara shows that in formal languages having finitary predicate and function symbols and in which "$\wedge$", "$\sim$", and "$\vee$" have their usual meanings and "$(\forall x)$" is equivalent to "$\sim(\exists x)\sim$" and, for some k, "$(\exists x)$" means "there exist at least ω_k elements x such that," the set of closed formulas which are true in all models of cardinality $\geqslant \omega_k$ is the same for each $k \geqslant 0$ and each corresponding interpretation of "$(\exists x)$". He calls this set of formulas **VI**. The set of closed formulas not in **VI** is called **SI**.

For each finite number n, "$(\exists x)$" can be interpreted to mean "there exist at least n elements x such that," and then the set of closed formulas true in all models having at least n elements is called $\mathbf{V}_n$. The set of closed formulas not in $\mathbf{V}_n$ is called $\mathbf{S}_n$. The intersection of all the sets $\mathbf{V}_n$ is called **VF**. If **V** is a set of formulas, then by **V**,2 we mean the set of formulas in **V** having only 2 quantifiers.

Our results are the following:

Theorem 1 $\mathbf{VF},2 \subsetneqq \mathbf{VI},2 \subsetneqq \mathbf{V}_1,2$.
Theorem 2 **VF**,2 *and* **VI**,2 *and* $\mathbf{V}_1$,2 *are recursive.*

Proof of Theorem 1: We first prove $\mathbf{VF},2 \subset \mathbf{VI},2$.

Case 1. If $(\exists x)(\forall y)P(x,y)$ is in **VF**,2, then it is in $\mathbf{V}_1$, by definition. So $(\forall x)(\exists y)\sim P(x,y)$ is not in $\mathbf{S}_1$ and therefore $\sim P(a_1,a_2) \wedge \sim P(a_2,a_3) \wedge \ldots \wedge \sim P(a_n,a_1)$ is, for all n, a quantifier-free formula which is not true under any valuation of its atomic formulas, because otherwise $\{a_1, a_2, \ldots, a_n\}$ would be the universe of a model for $(\forall x)(\exists y)\sim P(x,y)$. But this means that if "$(\exists x)$" is given the interpretation "there exist at least ω_0 elements x such that," then $(\forall x)(\exists y)\sim P(x,y)$ is unsatisfiable. Because if $\mathfrak{M}$ were a model for it, then there would be an element a_1 in $\mathfrak{M}$ such that there were infinitely many elements a_2 in $\mathfrak{M}$ such that $\mathfrak{M} \vdash \sim P(a_1,a_2)$. But all but a finite number of these elements a_2 would have infinitely many elements a_3 in $\mathfrak{M}$ such that $\mathfrak{M} \vdash \sim P(a_2,a_3)$. Thus we can find elements a_1, a_2, and a_3 in $\mathfrak{M}$ such that $\mathfrak{M} \vdash \sim P(a_1,a_2) \wedge \sim P(a_2,a_3)$.

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