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A SOLE SUFFICIENT OPERATOR

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Generations of students have been asked to prove (as an exercise) that the Sheffer stroke operator is a sole sufficient operator to define all of the monadic and dyadic operators in a two-valued space. A two-place functionally complete operator has come to be called a Sheffer operator [1]. We define a three-place operator S suggested by the work of A. A. Markov [2] in the theory of algorithms and prove that this operator is functionally complete over any finite-valued space. The proof is constructive.

Let $X(n)$ be the space of values $\mathsf{T} = 1, 2, \ldots, n = \mathsf{F}$. Over $X(n)$ define:

(1) $$Sxyz = \begin{cases} z, & \text{if } x = y; \\ x, & \text{if } x \neq y. \end{cases}$$

Consider, as an example, the two-valued case, $\mathsf{T} = 1, 2 = \mathsf{F}$. Negation, implication, conjunction, alternation, and the Sheffer stroke are defined by:

(2) $Nx = S\mathsf{T}x\mathsf{F}$; $Cxy = S\mathsf{T}xy$; $Kxy = Sx\mathsf{T}y$; $Axy = Sx\mathsf{F}y$; $Dxy = x/y = S\mathsf{T}Sx\mathsf{T}y\mathsf{F}$.

From this it is clear that S is a sole sufficient operator in the two-valued case.

In the general case we define n operators V_j, $1 \leqslant j \leqslant n$, such that V_jx has the value 1 if $x = j$, and V_jx has the value n if $x \neq j$.

(3) $$V_jx = \begin{cases} S1S1xnn, & \text{if } j = 1; \\ SSjx1jn, & \text{if } 2 \leqslant j \leqslant n. \end{cases}$$

If $x = j = 1$, $V_1 1 = S1S11nn = S1nn = 1$.
If $x \neq j = 1$, $V_1x = S1S1xnn = S11n = n$.
If $x = j \neq 1$, $V_jx = SSjj1jn = S1jn = 1$.
If $x \neq j \neq 1$, $V_jx = SSjx1jn = Sjjn = n$.

Hence definition (3) has the desired property. Define:

(4) $Kxy = Sx1y$.

Note that $K11 = 1$ and that $K1n = Kn1 = Knn = n$.

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