

A NOTE ON NATURAL DEDUCTION

ALEX BLUM

Theorem: *If S is a system of natural deduction which has an effective procedure for transforming its formulae into conjunctive normal form (from now 'cnf') and counts among its rules of inference Conjunction and*

$$\text{Tautology-Introduction (T-I): } \frac{\dots p \vee \dots \vee \sim p \dots}{\dots \sim p \vee \dots \vee p \dots}$$

*then S has a proof procedure for validity which is (transparently) effective.*¹

Proof: Let u be any valid formula in S . Let

$$u': a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n$$

be the expansion of u in cnf. We shall now prove u' valid.

1. a_1	T-I
2. a_2	T-I
3. a_3	T-I
⋮	
⋮	
n . a_n	T-I
$n+1$. $a_1 \cdot a_2 \cdot a_3 \cdot \dots \cdot a_n$	1, 2, 3, Conjunction

QED

1. The theorem also holds if in place of **T-I**, S contains: Conditional Proof, Addition, Tautology, Implication and Commutation as rules of inference.

Proof:

1'. p	assumption of limited scope
2'. $p \vee p$	1', Addition
3'. p	2', Tautology
4'. $p \supset p$	1'-3', Conditional Proof
5'. $\sim p \vee p$	4', Implication
6'. $\sim p \vee p \vee \dots$	5', Addition

Clearly, any inference sanctioned by **T-I** will be either 5' or a Commutation variant of either 5' or 6'.