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A STUDY OF $\mathcal{Z}$ MODAL SYSTEMS

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In [10], Sobociński has shown that the addition to various S4-extensions of Zeman's formula

Z1 $LMp \,.\, LMq \rightarrow (M(p \,.\, q) \rightarrow LM(p \,.\, q))$

generates a new family that he refers to as the $\mathcal{Z}$ modal systems. In this paper a completeness proof is given for the system Z1 = S4 + **Z1**, and the finite model property is established. Since the system is finitely axiomatisable its decidability follows. Furthermore it is shown that each $\mathcal{Z}$ modal system is the intersection of S5 with some system from family $\mathcal{K}$.

In the field of S4, **Z1** is inferentially equivalent to

Z2 $L(LMp \rightarrow MLp) \vee L(Mq \rightarrow LMq)$

the formula added to S4.4 by Schumm to obtain the system now called S4.9 (*cf.* Sobociński [9], p. 361)

(1)	$LM(p \vee q) \,.\, LM(\sim p \vee q) \rightarrow (M((p \vee q) \,.\, (\sim p \vee q)) \rightarrow LM((p \vee q) \,.\, (\sim p \vee q)))$	**Z1**, $p/p \vee q$, $q/\sim p \vee q$
(2)	$(p \vee q) \,.\, (\sim p \vee q) \leftrightarrow q$	**PC**
(3)	$Mq \rightarrow M((p \vee q) \,.\, (\sim p \vee q))$	(2), **C2**
(4)	$LM((p \vee q) \,.\, (\sim p \vee q)) \rightarrow LMq$	(2), **C2**
(5)	$LMp \rightarrow LM(p \vee q)$	**C2**
(6)	$\sim MLp \rightarrow LM(\sim p \vee q)$	**C2**
(7)	$LMp \,.\, \sim MLp \rightarrow (Mq \rightarrow LMq)$	(1), (3), (4), (5), (6), **PC**
(8)	$(LMp \rightarrow MLp) \vee (Mq \rightarrow LMq)$	(7), **PC**
(9)	$M(LMp \rightarrow MLp) \vee L(Mq \rightarrow LMq)$	(8), **C2**
(10)	$M(LMp \rightarrow MLp) \rightarrow (LMp \rightarrow MLp)$	S4
(11)	$(LMp \rightarrow MLp) \rightarrow L(LMp \rightarrow MLp)$	**Z1**, S4 (*cf.* [1])
Z2	$L(LMp \rightarrow MLp) \vee L(Mq \rightarrow LMq)$	(9), (10), (11), **PC**

This shows that Z1 contains the system S4 + **Z2**. We shall subsequently establish the converse in two different ways.

Definitions and discussion of the model-theoretic concepts used below are given in Segerberg [5], [6], and [7] (*cf.* also the Metatheorem of [1]).

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