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A NOTE ON NATURAL DEDUCTION IN MANY-VALUED LOGIC

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Natural-deduction techniques have not been applied very much in the formalization of many-valued logics, since the deduction theorem fails for many interesting systems; this point is made, for example, by Ackermann [1]. Nevertheless, natural-deduction formalizations are possible—an interesting example is Woodruff's [3]. In this note I describe a very simple natural-deduction system **Q** with rules in the style of Suppes [2]. The theorems of **Q** coincide with the theorems of **P**, which are the consequences under *modus ponens* of the axiom schemes

A1 $A \rightarrow (B \rightarrow A)$
A2 $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$
A3 $A \rightarrow ((A \rightarrow B) \rightarrow B)$

In **Q** the items in proofs are pairs $m\,A$ where m is a set, possibly empty, of positive integers and A is a formula. The numbers in m indicate the assumptions upon which A depends. The rules of **Q** are

R1 *For any formula A, the pair* (i) A *may be introduced at step number i in a proof.*
R2 *If m and n are disjoint, k B may be inferred from m A and n $(A \rightarrow B)$, k being the union of m and n.*
R3 *From* (i) A *occurring at step i and m B one may infer k $(A \rightarrow B)$, where k is the result of removing i from m.*

A is a theorem of **Q** if $\emptyset\,A$ is provable. $\emptyset$ is the null set. A1-A3 are easily shown to be theorems of **Q**:

1. (1) A **R1**
2. (2) B **R1**
3. (1) $B \rightarrow A$ 1, 2, **R3**
4. $\emptyset\, A \rightarrow (B \rightarrow A)$ 1,3, **R3**

Henceforth we omit $\emptyset$.

1. (1) $A \rightarrow B$ **R1**
2. (2) $B \rightarrow C$ **R1**

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