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PROPOSITIONAL AND PREDICATE CALCULUSES BASED ON COMBINATORY LOGIC

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In this article we shall establish various propositional and predicate calculuses based on combinatory logic (see [4]) with suitable restrictions on the variables. These restrictions are needed to avoid Curry's paradox ([4] pp. 258, 259). We require, Rule Ξ, the rule of restricted generality:

$$\Xi xy, xu \vdash yu,$$

and the iterated deduction theorem for Ξ.[1]

If $X_0, X_1, \ldots, X_m \vdash Y$ *where no* u_k *occurs in any* X_j *for* $j < k$, *and if for all* $k < m$ $X_0, X_1, \ldots, X_k \vdash \mathsf{L}([u_{k+1}]X_{k+1})$, *then* $X_0 \vdash X_1 \supset_{u_1} \ldots X_m \supset_{u_m} Y$.

From this deduction theorem we obtain deduction theorems for implication (P or $\supset$) and for universal generality (Π or $\Xi\mathsf{E}$).

If $X_0, X_1, \ldots, X_m \vdash Y$ *and if for all* $k < m$ $X_0, X_1, \ldots, X_k \vdash \mathsf{H}(X_{k+1})$ *then* $X_0 \vdash X_1 \supset : X_2 \supset \ldots \supset Y$.

If $X_0, X_1, \ldots X_m \vdash Yu$ *for all* u, *and* u *does not occur in any* X_j, *then* $X_0, X_1, \ldots X_m \vdash \Pi Y$.

The deduction theorem for Ξ was proved from certain axioms in [2], the other two are derived from it.

If we then take the additional axiom

Axiom PH. $\vdash \mathsf{H}x \supset_x . \mathsf{H}y \supset_y \mathsf{H}(x \supset y)$,

we obtain all of the absolute (or intuitionistic) calculus of pure implication. If we then introduce a slightly more complex axiom connecting Ξ and H,

1. In [2] L was defined as FAH or $\mathsf{B}(\Xi\mathsf{A}(\mathsf{BH}))$ and $\mathsf{H}X$ was interpreted as "X is a proposition." Here however we take L as primitive and define H by BLK. If we have $\mathsf{L} = \mathsf{FAH}$ and H primitive, we need either Axiom 2 or Axiom 8 ($\vdash\mathsf{LA}$) of [2] to prove Theorem 3 below. $Xu \supset_u Yu$ stands for ΞXY. Note that this form of the deduction theorem avoids the Kleene-Rosser paradox (See [3]).

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