INCIDENCE RINGS OF PRE-ORDERED SETS

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Introduction. In this paper* every relation $\leq$ on a set $X$ is a binary relation which is transitive and reflexive. G. C. Rota [2] has defined incidence rings of partially ordered systems $\langle X, \leq \rangle$. We generalize these rings by dropping the anti-symmetric condition on the order $\leq$.

If $X$ is a set and $\leq$ a binary relation on $X$, then $\langle X, \leq \rangle$ shall denote this relational system. We say that $\langle X, \leq \rangle$ is a pre-ordered relational system if the relation $\leq$ is transitive and reflexive. If confusion is unlikely, then we shall often take the liberty of using the relation $\leq$ to denote the usual ordering of the natural numbers and also to denote a relation on a set $X$. Unless otherwise stated, $0, 1$ should be understood to be real numbers. To each relational system $\langle X, \leq \rangle$ there is a unique zeta function, $\zeta$, mapping $X \times X$ into $\{0, 1\}$. For $x, y \in X$, $\zeta(x, y) = 1$, if $x \leq y$ and $\zeta(x, y) = 0$ otherwise. In the context of a relation system $\langle X, \leq \rangle$, $[x, y] = \{u \in X \mid x \leq u \leq y\}$ is an interval and $\langle X, \leq \rangle$ is locally finite iff every such interval is empty or a finite set.

We shall consider only rings $R$ which have a multiplicative identity; rings may or may not be commutative. We do not assume any relationship between the rings $R$ and sets $X$ we discuss. The symbol $R^*$ denotes the set of units of the ring $R$; the function $\det$ is the determinant function. If $n$ is a positive integer, then $M(n, R)$ denotes the complete ring of $n \times n$ matrices over the ring $R$. If $X$ is any set, then $S_X$ denotes the group of permutations of the set $X$; for positive integers $n$, $S_n$ denotes $S_{\{1, \ldots, n\}}$.

For a given ring $R$ and locally finite pre-ordered system $\langle X, \leq \rangle$, the incidence ring $I = \langle X, \leq, R \rangle$ is set—theoretically the set of functions $f$ mapping $X \times X$ into $R$ satisfying the following order condition. For every $x, y \in X$, $f(x, y) \neq 0$ only if $x \leq y$. Multiplication, addition and scalar multiplication for incidence rings are defined in section 1. If $[x, y]$ is a

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