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FURTHER EXTENSIONS OF S3*

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In [1] S3* was extended by 1. $\mathfrak{C}Kpq\mathfrak{C}\mathfrak{C}pqCLpLq$ to give S3** which is factorable in the sense of Zeman. By adding 2. $\mathfrak{C}LpLLp$ either to S3* or S3** we get of course into the area of S4. The weaker system should perhaps be chosen as S4*, the stronger one as S4**. Neither is factorable but if we add 3. $\mathfrak{C}KpLpLKpLp$ to S4** we obtain again a factorable system S4***. A still stronger system is given by adding 4. $\mathfrak{C}LpLKpLp$ to S3**. This we call $S4^{\triangle}$. It is obvious that we have:

$$\begin{array}{ccccccccc} S4 & \rightarrow & S4^{\triangle} & \rightarrow & S4^{***} & \rightarrow & S4^{**} & \rightarrow & S4^{*} \\ & & & & & & \downarrow & & \downarrow \\ & & & & & & S3^{**} & \rightarrow & S3^{*}. \end{array}$$

That the containments are proper is shown by the following matrices, to be taken with the usual Boolean four or eight valued matrices for C, N, K.

$\mathfrak{M}1$. L(*1*234) = 1333
$\mathfrak{M}2$. L(*1*234) = 1334
$\mathfrak{M}3$. L(*1*2*3*45678) = 15555778
$\mathfrak{M}4$. L(*1*234) = 2444
$\mathfrak{M}5$. L(*1*2*3*45678) = 15565556.

Then, $\mathfrak{C}Lpp$ is not in $S4^{\triangle}$ by $\mathfrak{M}1$; 4 is not in S4*** by $\mathfrak{M}2$; 3 is not in S4** by $\mathfrak{M}3$; 2 is not in S3** or S3* by $\mathfrak{M}4$; 1 is not in S3* or S4* by $\mathfrak{M}5$.

In the field of S3*, 5. $\mathfrak{C}KLpLqLKpq$ and 6. $\mathfrak{C}K\mathfrak{C}pq\mathfrak{C}qr\mathfrak{C}pr$ are inferentially equivalent. $\mathfrak{M}2$ shows that 5 is not in S4***, but it is not known whether it is in $S4^{\triangle}$. Assuming that it is not, then since {S3**, 5} evidently contains $S4^{\triangle}$ and by $\mathfrak{M}1$ lacks $\mathfrak{C}Lpp$, this system is properly intermediate between S4 and $S4^{\triangle}$. It can evidently be thought of as {$S4^0$, $CLpp$} and so should be called $R4^0$ on the analogy of Canty's R-systems in [2], but it should be noted that it lacks the rule to infer $L\alpha$ from α.

REFERENCES

[1] Thomas, Ivo, "Unusual feature of S3*," *Notre Dame Journal of Formal Logic*, vol. XIV (1973), p. 276.

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