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EXTENSIONS OF GÖDEL'S COMPLETENESS THEOREM AND THE LÖWENHEIM-SKOLEM THEOREM

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This note has a modest purpose: to present in the language of consequence operations some arguments, found in the theory of universal models, which extend the famous theorems of the title. This formulation provides further, but natural, motivations for the study of universal models besides the usual one, given for example in [3], chapter 10.

Let $\mathcal{L}$ be (the set of formulas of) a countable first-order language. A theory is a set of sentences of $\mathcal{L}$. Each theory T defines a consequence operation $\mathbf{C}_T$:

$$X \in \mathbf{C}_T(S) \textit{ iff } T \cup S \vdash X$$

where $X \in \mathcal{L}$, $S \subseteq \mathcal{L}$ and $\vdash$ is the standard provability relation. Each $\mathcal{L}$-structure $\mathfrak{A}$ also defines a consequence operation $\mathbf{C}_{\mathfrak{A}}$ as follows:

$$X \in \mathbf{C}_{\mathfrak{A}}(S) \textit{ iff } \forall h(\mathfrak{A} \models S[h] \Rightarrow \mathfrak{A} \models X[h])$$

where h ranges over all valuations of $\mathfrak{A}$ and where "$\mathfrak{A} \models S[h]$" means that h simultaneously satisfies all formulas in S. Note that $\mathbf{C}_{\mathfrak{A}}(\emptyset)$ is the set of formulas true in $\mathfrak{A}$.

A special case of Gödel's completeness theorem may now be written as follows:[1]

T is a complete[2] *theory iff* $\mathbf{C}_T(\emptyset) = \mathbf{C}_{\mathfrak{A}}(\emptyset)$, *for some structure* $\mathfrak{A}$.

The Löwenheim-Skolem theorem (for complete theories) becomes in this notation:

T is a complete theory iff $\mathbf{C}_T(\emptyset) = \mathbf{C}_{\mathfrak{A}}(\emptyset)$ *for some countable structure* $\mathfrak{A}$.

1. The general completeness theorem is:
$$\mathbf{C}_{\emptyset}(S) = \bigcap_{\mathfrak{A}} \mathbf{C}_{\mathfrak{A}}(S).$$
2. A complete theory is assumed consistent.

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