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A NEW PROOF OF COMPLETENESS

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We present a new proof of the completeness of the formalisation $\mathcal{P}$ of sentence logic based on the first four axioms of Russell's *Principia*, with substitution and modus ponens as rules of inference. For the sake of brevity we take for granted various elementary properties of $\mathcal{P}$, for instance that conjunction and disjunction are commutative and associative and that each distributes over the other; that $r \vee \neg r$ is provable in $\mathcal{P}$; that from $A \rightarrow P$ and $B \rightarrow P$ we may infer $(A \vee B) \rightarrow P$, and from $P \rightarrow A$, $P \rightarrow B$ we may infer $P \rightarrow (A \;\&\; B)$. It follows that if $\mathbf{T}$ denotes the provable sentence $r \vee \neg r$, and $\mathbf{F}$ denotes $\neg\mathbf{T}$ then the equivalences

$$p \longleftrightarrow (p \vee \mathbf{F}),\ \mathbf{T} \longleftrightarrow (p \vee \mathbf{T}),\ p \longleftrightarrow (p \;\&\; \mathbf{T})$$

are all provable in $\mathcal{P}$ from which it follows that

$$(*) \qquad p \longleftrightarrow (p \vee \mathbf{F}) \;\&\; (\neg p \vee \mathbf{T})$$

is provable in $\mathcal{P}$.

We start by observing that the negation of any one of the sentences of the set

$$p,\ \neg p,\ \mathbf{T},\ \mathbf{F}$$

and the disjunction of any two, is equivalent to a sentence of the set. It follows (by induction on the number of negations and disjunctions in a sentence) that any sentence $\mathfrak{S}(p)$ in the single variable p is equivalent to one of p, $\neg p$, $\mathbf{T}$, $\mathbf{F}$. Since

$$(p \vee \mathbf{T}) \;\&\; (\neg p \vee \mathbf{T}) \longleftrightarrow \mathbf{T}$$
$$(p \vee \mathbf{F}) \;\&\; (\neg p \vee \mathbf{F}) \longleftrightarrow \mathbf{F}$$
$$(p \vee \mathbf{T}) \;\&\; (\neg p \vee \mathbf{F}) \longleftrightarrow \neg p$$
$$(p \vee \mathbf{F}) \;\&\; (\neg p \vee \mathbf{T}) \longleftrightarrow p$$

are all provable, it follows that to each sentence $\mathfrak{S}(p)$ corresponds α, β such that

$$\mathfrak{S}(p) \longleftrightarrow (p \vee \alpha) \;\&\; (\neg p \vee \beta)$$

where each of α, β is one of $\mathbf{T}$, $\mathbf{F}$ (and so does not contain the variable p).

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