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VARIATIONS IN DEFINITION OF ULTRAPRODUCTS OF A FAMILY OF FIRST ORDER RELATIONAL STRUCTURES

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Two variations are made in the standard definitions, *cf.* [1], of an ultraproduct of a family of first order relational structures with respect to a chosen ultrafilter X of the index set I. The first variation, following a method used by W. A. J. Luxemburg, *cf.* [2] in the construction of higher order ultraproducts, relaxes the requirement of similarity on the members of the family. The second variation uses subfilters of X to define the individuals and relations of the ultraproduct.

In section **1** the construction of the ultraproduct with these variations is set out and some consequences developed, particularly those relating to the identity relation. In section **2** a family of similar structures is taken and a necessary and sufficient condition is established under which the first variation produces more relations, from an extensional view-point, than the standard definition.

1 Let $\{\mathbf{M}_i : i \in \mathsf{I}\}$ be a collection of first order relational structures. For each i, let $\mathbf{M}_i = \{R_i^0; R_i^1, R_i^2, \ldots\}$, where R_i^0 is the class (non empty) of individuals in the i^{th} structure and, for each positive integer k, R_i^k is the class of k-placed relations of the structure. Each R_i^k contains at least the empty relation and each R_i^2 contains the identity relation denoted by e_i. It is further assumed that the distinct members of each R_i^k are distinct from a set-theoretic and extensional point of view. Finally, if $a_1, \ldots, a_k \in R_i^0$ and $s^k \in R_i^k$ then "$s^k(a_1, \ldots, a_k)$" denotes the fact that $a_1, \ldots, a_k$ are related by s^k.

Let X be an ultrafilter defined on I. For each $k \geq 0$, X^k is a subfilter of X; that is X^k is a subclass of X and is a filter. For each $k \geq 0$, let R_{I}^k be the class $\{f^k : f^k : \mathsf{I} \to \bigcup\{R_i^k : i \in \mathsf{I}\}$ and for all $i \in \mathsf{I}, f^k(i) \in R_i^k\}$. Let $\sim_k$ denote the relation defined on R_{I}^k by: for all $f^k, g^k \in R_{\mathsf{I}}^k$, $f^k \sim_k g^k$ if, and only if, $\{i : f^k(i) = g^k(i)\} \in X^k$.

Lemma 1. *For each integer $k \geq 0$, $\sim_k$ is an equivalence relation.*

Proof: Immediate.

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