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EFFECTIVE INNER PRODUCT SPACES

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1 *Introduction* Dekker ([1]) introduced and studied a recursive enumerable vector space U_F over a recursive field F which is universal for all countable dimensional vector spaces over F. Many further results were gotten by Guhl [3], Metakides and Nerode ([7]), and others. The purpose of this paper* is to introduce a natural inner product on U_F and to show that the analogues of classical finite dimensional inner product space theory fail even for the recursive spaces.

2 *Preliminaries* We assume that the reader is familiar with the notations, conventions, and results of [1]. We let ε denote the set of nonnegative integers, and we note that 0 plays the role of both the Gödel number of the zero element of F and the zero vector of U_F. If β is a repère (a linearly independent set) in U_F and x is a member of $\mathbf{L}(\beta)$, we write $\mathrm{supp}_\beta(x)$ for the set of all elements of β which have nonzero coefficients when x is expressed as a linear combination of elements in β. We let $\eta = \rho e$ be the canonical basis for U_F and write $\mathrm{supp}(x)$ for $\mathrm{supp}_\eta(x)$. Following [8], Chapter 11, we call the field F *formally real* if -1_F is not expressible in F as a sum of squares. Note that F is formally real if and only if a sum of squares of elements of F vanishes only when each element is zero. All formally real fields have characteristic 0; Q, $Q(\sqrt{2})$, $Q(\pi)$ are formally real, while $Q(\mathrm{i})$ is not.

Definition D1: Let F be any countable formally real field for which there exists a one-to-one mapping ϕ from F onto ε under which the field operations correspond to (partial) recursive functions. We consider the recursively presented vector space U_F over F constructed in [1]. We define a function $\langle , \rangle$ from $\varepsilon \times \varepsilon \rightarrow \varepsilon$ by

$$\langle x, y\rangle = \phi\left(\sum_{i=1}^{k} x_i y_i\right) ,$$

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