Notre Dame Journal of Formal Logic
Volume XIX, Number 4, October 1978
NDJFAM

THE IDEAL OF ORDERABLE SUBSETS OF A SET

JOHN L. HICKMAN

The Ordering Principle (**OP**),* which states that every set can be linearly ordered, is not provable in Zermelo-Fraenkel (**ZF**) set theory. Let $O(S)$ be the set of all orderable subsets of a set S: obviously $O(S)$ will be of intrinsic interest only when S cannot be ordered, that is, only when $O(S) \neq P(S)$, the power-set of S. If $X, Y \in O(S)$, then $X \cup Y \in O(S)$; and if $X \in O(S)$ and $Y \subseteq X$, then $Y \in O(S)$. Thus $O(S)$ is an ideal of $P(S)$ (regarded as the usual algebra), and clearly $O(S) \supseteq P^{\circ}(S)$, where $P^{\circ}(S)$ is the set (ideal) of all finite subsets of S. We therefore have two extreme possibilities (when S is non-orderable): (i) the lower extreme, when $O(S) = P^{\circ}(S)$; and (ii) the upper extreme, when $O(S)$ is a maximal ideal. Both these extremes, as well as varying positions in between, can be attained.

Definition 1

(i) A set S is called "finite" if there is a bijection $f: n \to S$ for some natural number n.
(ii) A set S is called "Dedekind-finite" if there is no $T \subseteq S$ with $T \neq S$ for which there is a bijection $f: S \to T$.
(iii) A set S is called "medial" if S is infinite and Dedekind-finite.
(iv) A set S is called "quasi-minimal" if for every $X \subseteq S$, exactly one of X, $S - X$ is finite.

In (i) above, we are regarding natural numbers (and ordinals in general) to be defined in such a way that each is the set of all smaller natural numbers (ordinals). It is easily seen that every quasi-minimal (**qm**) set is medial, and that a set S is Dedekind-finite if and only if there is no injection $f: \omega \to S$. Concerning the existence of **qm** sets, we refer to §1 of [1].

Lemma 1 *If S is quasi-minimal, then $P(S)$ is Dedekind-finite.*

*The work contained in this paper was done whilst the author held a Research Fellowship at the Australian National University.

Received July 17, 1976