Notre Dame Journal of Formal Logic
Volume XIX, Number 4, October 1978
NDJFAM

PROVABLY RECURSIVE REAL NUMBERS

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1 *Introduction* In this paper we shall begin development of a theory of provably recursive real numbers similar in spirit to the theory of provable recursive functions discussed by Kreisel [7, 8], Fischer [4] and Ritchie and Young [12]. For example, we say that a program i names a provably recursive real number if we can prove (in some axiomatization at least as powerful as elementary number theory) that the function defined by i is total and satisfies a recursive Cauchy criterion. Of special interest will be the contrasts between provably recursive real numbers and recursive real numbers.

Neither our base theory nor our metatheory will be specified explicitly. For our base theory we let **S** be an axiomatization of any theory which encompasses elementary number theory; we require that our metatheory be powerful enough to express the soundness of **S** for arithmetic. All of our theorems will have as an implicit hypothesis that **S** is sound for arithmetic; this hypothesis would be unnecessary if **S** were an axiomatization of elementary number theory and the metatheory were full set theory because we can prove in set theory that **S** is sound for arithmetic. We take as fixed the enumeration $\phi_0, \phi_1, \phi_2, \ldots$ of partial recursive functions of one variable described in Davis [3] (which can be proven, in **S**, to be a standard enumeration). Let $K = \{i: \phi_i(i)\downarrow\}$. K is a nonrecursive, recursively enumerable set (and a proof of this can be carried out in **S**). Let rat (see Kleene and Post [6]) be a one-to-one primitive recursive function mapping the set N of natural numbers onto the set Q of rationals in lowest terms.

Our treatment will be informal: we will usually present the intuitive idea behind a proof and leave to the reader the details of carrying the proof

*This work includes results from the author's Ph.D. thesis and was partially supported by NSF Research Contract GJ-271-27-A1 at Purdue University, and partially supported by a research grant from Hofstra University.

Received October 25, 1974